

Singularities on the World Sheets of Open Relativistic Strings

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Abstract

A classification of stable singular points on world sheets of open relativistic strings is carried out.

1. Introduction

String theory considers surfaces in d -dimensional Minkowski space, which have an extreme area – world sheets of string. Analogous theory in Euclidean space studies soap films. String theory considers the world sheets of various topological types: open strings (obtained by a mapping of a band into Minkowski space), closed strings (mapping of a cylinder), Y-shaped strings (mapping of 3 bands, glued together along one edge), etc. A finer classification takes into account the smoothness of these mappings. It is known that world sheets can have stable singular points. An important role of singularities in string theory was mentioned by many authors.

- Singularities have a form of angular points on string. Density of string's energy-momentum tends to infinity in a singular point. Near singular points string models of hadrons go beyond the limits of their applicability. String breaking may occur in singular points [1].
- Singularities on the world sheets in 3D space behave like point particles, that move at the velocity of light, scatter in collisions and can form bound states. In σ -model approach to string theory the singularities manifest themselves as singular solutions of Liouville equation, they form a completely integrable Hamiltonian system [2, 3]. Physical interpretation of singularities on string as the elements of exotic hadrons structure is very attractive, for example, as valent gluons [4]. This identification allows one to describe exotic hadrons states in the frames of string model.
- Singularities conserve in deformations of the world sheet. In approaches to string quantization at non-critical dimension [5] their number plays a role of super-conserving value (topological charge), dividing the phase space

into disconnected sectors. For the sector with zero topological charge the self-consistent quantum theory can be constructed.

As far as we know, a detailed classification of singular points on the world sheets of strings has not been developed yet. The main goal of this work is a comprehensive study of singularities on the world sheets of open strings. We have developed a set of visualization tools [6] for the complicated three-dimensional surfaces drawing and used it in this study. We have confirmed the results of earlier investigations of singularities on world sheets [2, 7] and also observed new phenomena.

2. World sheet of open string

Let a periodical isotropic curve $Q_\mu(\sigma)$ be given in d -dimensional Minkowski space:

$$Q'^2(\sigma) = 0, \quad Q_\mu(\sigma + 2\pi) = Q_\mu(\sigma) + 2P_\mu,$$

$2P_\mu$ is a period of the curve. This curve is shown on fig.1. We will refer to this curve as the supporting curve.

The world sheet of open string is constructed as geometrical place of middles of segments, connecting all possible pairs of points on supporting curve [8]

$$x_\mu(\sigma_1, \sigma_2) = \frac{1}{2} (Q_\mu(\sigma_1) + Q_\mu(\sigma_2)). \quad (1)$$

Let's prove this. The action of string is proportional to an area of the world sheet

$$A = -\frac{1}{2\pi\alpha'} \int \mathcal{L} d\sigma_1 d\sigma_2, \quad \mathcal{L} = \sqrt{-\frac{1}{2} \mathcal{L}^{\mu\nu} \mathcal{L}_{\mu\nu}},$$

$$\mathcal{L}^{\mu\nu} = \epsilon_{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \quad \alpha, \beta = 1, 2$$

(further we fix a system of units $2\pi\alpha' = 1$).

The condition for an extremum of action has a form of continuity equation

$$\partial_\alpha p_\alpha^\mu = 0, \quad p_\alpha^\mu = \frac{\delta A}{\delta \partial_\alpha x^\mu} = \frac{\mathcal{L}^{\mu\nu}}{\mathcal{L}} \epsilon_{\alpha\beta} \partial_\beta x^\nu, \quad (2)$$

p_α^μ is momentum density, flowing through a section $\sigma_\alpha = Const.$

The integral form of this condition

$$\oint_C p_\alpha^\mu \epsilon_{\alpha\beta} d\sigma_\beta = 0 \quad (3)$$

has a sense of momentum conservation law: total momentum flowing through a closed contour C on the world sheet, vanishes.

In the Euclidean space analogous equations give a shape of the soap film surface. In the theory of soap films the area is proportional to potential energy, derivatives (2) are equal to forces of the surface tension. Equation (3) describes an equilibrium of a part of the surface, restricted by the contour C : the total force of tension, acting on this part, vanishes.

One can easily check by substitution, that surface (1) obeys condition (2):

$$\partial_\alpha x^\mu = \frac{1}{2} Q'^\mu(\sigma_\alpha), \quad \mathcal{L}^{\mu\nu} = \frac{1}{4} (Q'^\mu(\sigma_1)Q'^\nu(\sigma_2) - Q'^\mu(\sigma_2)Q'^\nu(\sigma_1)),$$

$$\mathcal{L} = \frac{1}{4} (Q'(\sigma_1)Q'(\sigma_2)), \quad p_1^\mu = -\frac{1}{2} Q'^\mu(\sigma_2), \quad p_2^\mu = -\frac{1}{2} Q'^\mu(\sigma_1) \quad \Rightarrow \quad \partial_\alpha p_\alpha^\mu = 0.$$

Surface (1) has two edges (fig.1). The first edge coincides with the supporting curve

$$x_\mu(\sigma, \sigma) = Q_\mu(\sigma).$$

The second edge is obtained from the supporting curve by translation onto a semi-period:

$$x_\mu(\sigma + 2\pi, \sigma) = Q_\mu(\sigma) + P_\mu.$$

The world sheet transforms into itself in translation on $2P_\mu$. When translated on P_μ it also transforms into itself, in this its edges interchange.

Besides the local equation (2), from the condition of minimum of action it follows that the momentum flow through the edge vanishes. This requirement holds:

$$\int_{\text{along the edge}} p_\alpha^\mu \epsilon_{\alpha\beta} d\sigma_\beta = \frac{1}{2} \int_{(\sigma, \sigma)}^{(\sigma', \sigma')} Q'^\mu(\sigma_1) d\sigma_1 - Q'^\mu(\sigma_2) d\sigma_2 = 0.$$

The edges of the world sheet are isotropic: $Q'^2 = 0$. Besides, any vector from the tangent plane to the world sheet on its edge is orthogonal to Q'_μ . This plane is tangent to the light cone in the direction of Q'_μ . We will call these planes the isotropic planes.

Proof. Parametrization of world sheet (1) is non-regular on the edge, because tangent vectors $\frac{\partial x_\mu}{\partial \sigma_1}$ and $\frac{\partial x_\mu}{\partial \sigma_2}$ are linearly dependent when $\sigma_1 = \sigma_2 = \sigma_0$

$$\frac{\partial x_\mu}{\partial \sigma_1} = \frac{\partial x_\mu}{\partial \sigma_2} = \frac{1}{2} Q'_\mu(\sigma_0).$$

Let's expand $x_\mu(\sigma_1, \sigma_2)$ in the vicinity of point (σ_0, σ_0) :

$$\begin{aligned} x^\mu(\sigma_0 + \Delta\sigma_1, \sigma_0 + \Delta\sigma_2) &= Q_0^\mu + \frac{1}{2} Q_0'^\mu \cdot (\Delta\sigma_1 + \Delta\sigma_2) + \\ &+ \frac{1}{4} Q_0''^\mu \cdot (\Delta\sigma_1^2 + \Delta\sigma_2^2) + o(\Delta\sigma_1^2 + \Delta\sigma_2^2). \end{aligned}$$

Regular parametrization in the vicinity of this point is given by

$$\rho_1 = \Delta\sigma_1 + \Delta\sigma_2, \quad \rho_2 = \Delta\sigma_1^2 + \Delta\sigma_2^2.$$

Basis vectors of the tangent plane are Q'_0 and Q''_0 , $Q'_0 Q''_0 = \frac{1}{2} (Q'^2)'_0 = 0$, then all vectors from the tangent plane are orthogonal to Q'_0 . Equation of $(d-1)$ -dimensional tangent space to light cone $Q'^2 = 0$ in the point Q'_0 has a form: $dQ'^2|_0 = 2Q'_0 dQ' = 0$, i.e. vectors dQ' from tangent space are orthogonal to Q'_0 . The plane specified by vectors Q'_0 and Q''_0 is contained in the tangent space to light cone (if $d = 3$, these planes coincide).

Because $\rho_2 \geq 0$, then up to terms $o(\rho_2)$ the world sheet lies in a semi-plane of tangent plane, bounded by the straight line in the direction of Q'_0 and containing vector Q''_0 . Therefore, the supporting curve is the edge of the world sheet.

The edges of world sheet are world lines of string ends. Slice of the world sheet by constant time plane gives the string at this time moment. The tangent to the string l_μ is contained in the tangent plane to the world sheet, that's why on the edge it is orthogonal to the tangent to the edge

$$l_\mu Q'_\mu = 0 \quad \Rightarrow \quad (l_0 = 0) \quad \vec{l} \vec{Q}' = 0.$$

Therefore, the ends of the string move with light velocity at right angle to the direction of string in this point.

Total momentum of the string

$$\int_C p^\mu_{\alpha} \epsilon_{\alpha\beta} d\sigma_\beta = \frac{1}{2} \int_{(\sigma,\sigma)}^{(\sigma+2\pi,\sigma)} Q'^\mu(\sigma_1) d\sigma_1 - Q'^\mu(\sigma_2) d\sigma_2 = P_\mu$$

is equal to semi-period of the supporting curve (we have calculated an integral along a contour, connecting the edges of the sheet).

The projection of the supporting curve into a subspace, orthogonal to P_μ (the rest frame of string) is the closed curve $\vec{Q}(\sigma)$. The supporting curve may be restored by its projection:

$$Q'_0 = |\vec{Q}'| \quad \Rightarrow \quad Q_0(\sigma_1) - Q_0(0) = \int_0^{\sigma_1} |\vec{Q}'(\sigma)| d\sigma = L(\sigma_1),$$

$L(\sigma)$ is the arc length of the curve $\vec{Q}(\sigma)$ between the points $\vec{Q}(0)$ and $\vec{Q}(\sigma)$. The total length of the curve $\vec{Q}(\sigma)$ is equal to double mass of string: $L(2\pi) = 2\sqrt{P^2}$. One can parametrize the curve $\vec{Q}(\sigma)$ by its length: $\sigma = \pi L / \sqrt{P^2}$, then $Q'_0 = |\vec{Q}'| = \sqrt{P^2} / \pi$. We will refer to this parametrization as the rest frame gauge (RFG).

The shape of string is determined in RFG in the following way [4]. Let's choose an arbitrary point A on the curve $\vec{Q}(\sigma)$ and draw two arcs of equal

length along the supporting curve in the opposite sides from point A and mark the middle of the segment, connecting these points. A set of all middles forms a curve. This curve is the string at time moment, when its end is placed in point A.

In string theory a parametrization is often used

$$\sigma = \pi \frac{(nQ)}{(nP)}, \quad n_\mu \text{ is a constant isotropic vector,}$$

which is called a light cone gauge (LCG). It is convenient to resolve an equation $Q'^2 = 0$:

$$n = (1, 1, \dots, 0), \quad Q_\pm = Q_0 \pm Q_1, \quad \vec{Q}_\perp = (Q_2, \dots, Q_{d-1}),$$

$$Q'^2 = Q'_+ Q'_- - \vec{Q}'_\perp{}^2 = 0 \quad \Rightarrow \quad Q_- = \frac{\sigma}{\pi} P_-, \quad Q_+ = \frac{\pi}{P_-} \int \vec{Q}'_\perp{}^2(\sigma) d\sigma.$$

If transverse components $\vec{Q}_\perp(\sigma)$ are given in the form of Fourier expansion, then the last integration can be developed explicitly.

Nevertheless, it occurs that the light cone gauge “does not work” in space $d = 2 + 1$. We will return to this question below.

The world sheet (1) can have singularities. Point of the surface can be singular, if tangent vectors $\frac{\partial x_\mu}{\partial \sigma_1}$ and $\frac{\partial x_\mu}{\partial \sigma_2}$ are linearly dependent in it:

$$\frac{\partial x_\mu}{\partial \sigma_1} = \frac{1}{2} Q'_\mu(\sigma_1) \quad \parallel \quad \frac{\partial x_\mu}{\partial \sigma_2} = \frac{1}{2} Q'_\mu(\sigma_2).$$

Therefore, for the appearance of singularities on the world sheet it is necessary, that the supporting curve has two points with parallel tangents (fig.2).

Tangents are parallel in the points of the supporting curve, separated by the period. We have found that these points correspond to the edge of the world sheet. The edge is the singular line on the world sheet. A surface, obtained by mapping (1) of the whole plane of parameters (σ_1, σ_2) into Minkowski space, has many coincident sheets – it repeatedly covers itself, each point of it is covered with infinite multiplicity. One can easily get convinced in this, checking that transformations

$$\sigma_1 \leftrightarrow \sigma_2; \quad \sigma_1 \rightarrow \sigma_1 + 2\pi n, \quad \sigma_2 \rightarrow \sigma_2 - 2\pi n$$

do not change the point on the surface. In fact, the edge of the surface is a fold, on which different sheets of this surface are linked. Physically only one sheet represents the string, it can be extracted by the restriction of parameters (σ_1, σ_2) in a band $\sigma_1 \leq \sigma_2 \leq \sigma_1 + 2\pi$.

A surface, consisting of k of such sheets, also represents a possible evolution of string – it is a degenerate case of k -folded string [5]. Degeneration is removed, if one introduces a small deformation on each k -th period of the supporting curve. In this $2P_\mu$ -periodical curve becomes $2kP_\mu$ -periodical, and pleats of k coincident sheets become disjointed.

We will look for points with parallel tangents inside one period of the supporting curve. In RFG this parallelism condition takes form of equality of unit tangent vectors to the projection of the supporting curve $\vec{Q}(\sigma)$:

$$\vec{Q}'(\sigma_1) = \vec{Q}'(\sigma_2), \quad Q'_0(\sigma_1) = Q'_0(\sigma_2) = \sqrt{P^2}/\pi.$$

Singularities on the world sheet in spaces $d = 3$ and $d = 4$ have different shapes.

3. Singularities on world sheets in $d = 3$ – kink lines.

When $d = 3$, a space orthogonal to the momentum P_μ is two-dimensional. Smooth closed curves on a plane are divided into classes with respect to the winding number ν of the unit tangent vector. For convex curves (which do not have inflection points) a polar angle of the unit tangent vector depends on σ monotonously. For each direction of tangent $\nu - 1$ points will be found which have the same tangent direction. Therefore, $\nu - 1$ singular points are ever existing on the string in $d = 3$. Singular points on the world sheet form lines. Tangents to them are isotropic:

$$dx_\mu = \frac{1}{2} (Q'_\mu(\sigma_1)d\sigma_1 + Q'_\mu(\sigma_2)d\sigma_2) \parallel Q'_\mu(\sigma_1).$$

Let's study a shape of world sheet in the vicinity of a singular point.

$$\begin{aligned} x(\sigma_1 + \Delta\sigma_1, \sigma_2 + \Delta\sigma_2) &= \frac{1}{2} (Q_1 + Q_2) + \frac{1}{2} Q'_1 \cdot (\Delta\sigma_1 + \lambda\Delta\sigma_2) + \\ &+ \frac{1}{4} (Q''_1\Delta\sigma_1^2 + Q''_2\Delta\sigma_2^2) + o(\Delta\sigma_1^2 + \Delta\sigma_2^2), \\ Q'_2 &= \lambda Q'_1. \end{aligned} \quad (4)$$

Vectors Q'_1, Q''_1, Q''_2 are linearly dependent, because Q'_1 and Q''_2 lie in 2-dimensional plane, orthogonal to Q'_1 , and this plane contains Q'_1 .

$$Q''_2 = \alpha Q'_1 + \beta Q''_1,$$

$$\begin{aligned} x &= \frac{1}{2} (Q_1 + Q_2) + \frac{1}{2} Q'_1 \cdot (\Delta\sigma_1 + \lambda\Delta\sigma_2 + \frac{\alpha}{2}\Delta\sigma_2^2) + \\ &+ \frac{1}{4} Q''_1 \cdot (\Delta\sigma_1^2 + \beta\Delta\sigma_2^2) + o(\Delta\sigma_1^2 + \Delta\sigma_2^2). \end{aligned} \quad (5)$$

Mapping $(\Delta\sigma_1, \Delta\sigma_2) \rightarrow (X, Y) = (\Delta\sigma_1 + \lambda\Delta\sigma_2, \Delta\sigma_1^2 + \beta\Delta\sigma_2^2)$ has a fold on parabola $Y = X^2/(1 + \lambda^2/\beta)$:

$$\begin{aligned} \det \frac{\partial(X, Y)}{\partial(\Delta\sigma_1, \Delta\sigma_2)} &= 2(\beta\Delta\sigma_2 - \lambda\Delta\sigma_1) = 0 \quad \Rightarrow \quad \Delta\sigma_2 = \frac{\lambda}{\beta}\Delta\sigma_1, \\ X &= (1 + \frac{\lambda^2}{\beta})\Delta\sigma_1, \quad Y = (1 + \frac{\lambda^2}{\beta})\Delta\sigma_1^2. \end{aligned}$$

An image of plane $(\Delta\sigma_1, \Delta\sigma_2)$ lies over parabola at $\beta > -\lambda^2$ and under parabola at $\beta < -\lambda^2$:

$$(1, 0) \rightarrow (1, 1), \quad 1 > 1/(1 + \lambda^2/\beta) \quad \text{when} \quad \beta > -\lambda^2.$$

In a small vicinity of the point of expansion the world sheet is close to a semi-plane with a boundary, parallel to Q'_1 , containing vector $Q''_1 \operatorname{sgn}(\beta + \lambda^2)$. Therefore, when passing through the singular line, the world sheet undergoes a fold (fig.3). Parts of the world sheet, separated by the singular line, have common tangent plane near the singular line. This plane is isotropic, as the tangent plane on the edge.

Slice of the world sheet by constant time plane has a cusp. Such objects on the string are called kinks. The direction of the tangent to string in the cusp (kink direction) is orthogonal to the kink's world line, because the tangent plane to the world sheet on the kink line is isotropic. The kink moves with light velocity at the right angle to the string direction in this point.

We give a physical explanation for such behaviour of ends and kinks. We will show that a mass of vanishingly small part of string adjacent to its end tends to zero faster than a force of tension acting on this part.

A mass of the small part of string is proportional to its length:

$$\begin{aligned} dx &= \frac{1}{2} (Q'_1 d\sigma_1 + Q'_2 d\sigma_2) \quad (dx^0 = 0), \\ dp &= \frac{1}{2} (Q'_1 d\sigma_1 - Q'_2 d\sigma_2), \\ (dp)^2 &= -\frac{1}{2} (Q'_1 Q'_2) d\sigma_1 d\sigma_2 = -(dx)^2 = dl^2. \end{aligned}$$

Let's consider the part of length Δl , adjacent to the end of string (σ_0, σ_0) :

$$\begin{aligned} \Delta l &= \int_{\sigma_0}^{\sigma_0 + \Delta\sigma_1} \sqrt{\frac{1}{2} (Q'_1 Q'_2) \frac{Q_1^{0'}}{Q_2^{0'}}} d\sigma_1 = \frac{1}{2} \sqrt{-Q_0'^2} \Delta\sigma_1^2, \\ \Delta\sigma_1 &= -\Delta\sigma_2 = \sqrt{\frac{2\Delta l}{\sqrt{-Q_0'^2}}}. \end{aligned}$$

The following relations are used

$$\begin{aligned} dx^0 = 0 &\Rightarrow d\sigma_2 = -\frac{Q_1^{0'}}{Q_2^{0'}} d\sigma_1, \\ (Q' Q''') + Q''^2 &= (Q' Q'')' = 0, \\ (Q'_1 Q'_2) &= \left(Q'_0 + Q_0'' \Delta\sigma_1 + \frac{1}{2} Q_0''' \Delta\sigma_1^2 + o(\Delta\sigma_1^2), \quad Q'_0 + Q_0'' \Delta\sigma_2 + \right. \\ &\quad \left. + \frac{1}{2} Q_0''' \Delta\sigma_2^2 + o(\Delta\sigma_2^2) \right) = -2Q_0''^2 \Delta\sigma_1^2 + o(\Delta\sigma_1^2) \end{aligned}$$

Let's obtain a force acting on this part:

$$\begin{aligned}
\Delta p &= \frac{1}{2} \int_{(\sigma_0 + \Delta\sigma_1, \sigma_0 - \Delta\sigma_1)}^{(\tilde{\sigma}_0 + \Delta\tilde{\sigma}_1, \tilde{\sigma}_0 - \Delta\tilde{\sigma}_1)} Q'_1 d\sigma_1 - Q'_2 d\sigma_2 = \\
&= \frac{1}{2} \left(Q(\tilde{\sigma}_0 + \Delta\tilde{\sigma}_1) - Q(\sigma_0 + \Delta\sigma_1) - Q(\tilde{\sigma}_0 - \Delta\tilde{\sigma}_1) + Q(\sigma_0 - \Delta\sigma_1) \right) = \\
&= Q'(\tilde{\sigma}_0)\Delta\tilde{\sigma}_1 - Q'(\sigma_0)\Delta\sigma_1 = \left(\frac{Q'}{(-Q''^2)^{1/4}} \right)'_{\sigma=\sigma_0} \sqrt{2\Delta l} (\tilde{\sigma}_0 - \sigma_0) , \\
\Delta t &= Q_0^{0'} (\tilde{\sigma}_0 - \sigma_0) , \\
F &= \frac{\Delta p}{\Delta t} \sim \sqrt{\Delta l}, \quad m \sim \Delta l \quad \text{then} \quad F/m \sim 1/\sqrt{\Delta l} \rightarrow \infty.
\end{aligned}$$

Under such conditions a relativistic particle m moves with light velocity.

The precision of expansions used is insufficient in the points with zero curvature $Q''_0 = 0$. In work [1] an interesting phenomenon was observed. Let's consider the supporting curve with straight line segment $Q'_1 = Q'_2 \equiv Q'$. For this curve

$$dx^0 = 0 \Rightarrow d\sigma_1 = -d\sigma_2, \quad d\vec{x} = 0, \quad dp = Q' d\sigma_1 ,$$

when $d\sigma_1$ changes in finite interval. Therefore, a finite isotropic momentum is concentrated in one point at the end of string. This momentum accumulates when the end of string passes along the straight segment and then flows away up to a moment, when the end of string leaves the straight segment.

Proof for the kink is analogous. An essential difference between the kink line and the edge of the sheet is that the momentum flow through the kink line does not vanish. For parts of the world sheet, separated by the kink line, momenta do not conserve separately. These parts are not minimal surfaces. We will show that these parts are causally linked in spite of the fact that metric on the world sheet degenerates on the kink line, and all tangent directions to world sheet on kink line are space-like, except single isotropic direction along kink line.

More about kink direction

Value β/λ^2 is the parametric invariant:

$$\begin{aligned}
\sigma \rightarrow f(\sigma) &\Rightarrow Q' \rightarrow Q' f', \quad Q'' \rightarrow Q'' f'^2 + Q' f'', \\
\lambda = \frac{Q_2^{0'}}{Q_1^{0'}} &\rightarrow \lambda f'_2 / f'_1, \quad \beta = \frac{(Q_1'' Q_2'')}{Q_1''^2} \rightarrow \beta (f'_2 / f'_1)^2.
\end{aligned}$$

In RFG $Q^{0''} = 0$, $\lambda = 1$, $\beta = (\vec{Q}_1'' \vec{Q}_2'') / \vec{Q}_1''^2$. Since $\vec{Q}'' \perp \vec{Q}'$ and $\vec{Q}_1' = \vec{Q}_2'$, then $\vec{Q}_1'' \parallel \vec{Q}_2''$. Therefore $\beta = \pm \frac{|\vec{Q}_2''|}{|\vec{Q}_1''|}$ (+, if $\vec{Q}_1'' \uparrow \vec{Q}_2''$; -, if $\vec{Q}_1'' \downarrow \vec{Q}_2''$). $|\vec{Q}''|$ is proportional to a curvature of the curve $\vec{Q}(\sigma)$: $|\vec{Q}''| = \left| \frac{d\varphi}{d\sigma} \right| \frac{\sqrt{P^2}}{\pi}$, where φ is the polar angle of the tangent vector \vec{Q}' . If vectors \vec{Q}_1'' and \vec{Q}_2'' have the same direction, then kink is directed along them. If \vec{Q}_1'' and \vec{Q}_2'' have the opposite directions, then the kink is directed along the vector of a greater module. Note, that for convex curves $\vec{Q}(\sigma)$ vectors \vec{Q}_1'' and \vec{Q}_2'' have the same direction, because orientation of the pair \vec{Q}', \vec{Q}'' conserves on these curves. This orientation is connected with orientation of the supporting curve: the direction of \vec{Q}'' is obtained by rotation of \vec{Q}' through $\pi/2$ in the direction of rotation of tangent (fig.4).

For the convex supporting curves the kinks move inside the string never reaching the edges. (The polar angle of tangent vector monotonously changes with σ , its rate $\frac{d\varphi}{d\sigma}$ is bounded from above and from below, if the curvature of the supporting curve never becomes zero or infinity. Therefore, points with parallel tangents cannot be close as one wishes on the supporting curve, a “distance” $\Delta\sigma$ between such points is separated from zero by positive number.)

The world sheet for the supporting curve fig.4 is viewed in fig.5.

Density of lines on the left bottom part of this figure is proportional to time averaged density of the string mass distribution. Points, where density sharply changes, are placed on the kink line and roundings of strings.

Non-convex supporting curves

For non-convex supporting curves the direction of rotation of tangent changes in inflection points. In this new pairs of points with parallel tangents appear on the curve, new kinks appear on string (fig.6).

The number of inflection points on closed smooth curves is even. Let's consider the case, when the supporting curve has two inflection points.

Fig.7 displays the polar angle of tangent vector versus σ . On the segments a, b, c for any point σ_1 a point σ_2 exists, which has the same tangent. These points form curves on the plane of parameters (σ_1, σ_2) :

$$ab = \{(\sigma_1, \sigma_2) : \varphi(\sigma_1) = \varphi(\sigma_2), \sigma_1 \in a, \sigma_2 \in b\} \text{ etc.}$$

Let us consider some details of fig.7. Consider the sections of a graph $\varphi(\sigma)$ by line $\varphi = \text{Const}$, moving down. Let $\sigma_1 < \sigma_2$. The equation $\varphi(\sigma_1) = \varphi(\sigma_2)$ begins to have solutions when $\varphi < \varphi_A$. At this moment from the point A the line ab originates. In the vicinity of the point A

$$\varphi(\sigma_A + \Delta\sigma) = \varphi_A + \frac{1}{2} \varphi_A'' \Delta\sigma^2 + o(\Delta\sigma^2),$$

the solution of equation $\varphi(\sigma_1) = \varphi(\sigma_2)$ in low approximation is $\sigma_1 = \sigma_A + \Delta\sigma$, $\sigma_2 = \sigma_A - \Delta\sigma$. Therefore near point A the line ab is orthogonal to the line $\sigma_1 = \sigma_2$. In further descending of φ the value σ_1 decreases and σ_2 increases until φ becomes equal to φ_B . At this moment a rate of σ_2 increases without bound, the rate of σ_1 remains bounded (the rate of φ is constant), therefore near $\varphi = \varphi_B$ the tangent to ab is vertical. In this $\sigma_2 = \sigma_B$. Now let's lift φ . The value σ_2 passes from interval b to interval c . On the line ac values σ_1 and σ_2 increase until φ reaches φ_A . At this moment σ_1 has infinite rate, the tangent to ac is horizontal. In this $\sigma_1 = \sigma_A$. Next we again decrease φ . The value σ_1 passes to interval b . The line bc terminates in the point B .

Replacement $\sigma_1 \leftrightarrow \sigma_2$ gives the whole picture.

In RFG $\sigma \sim Q^0$, points of world sheet, satisfying $\sigma_1 + \sigma_2 = Const$, are at the same time in the rest frame. Value β represents a slope of the tangent to the kink line on the plane (σ_1, σ_2) :

$$\vec{Q}'(\sigma_1) = \vec{Q}'(\sigma_2) \Rightarrow \vec{Q}''(\sigma_1) \frac{d\sigma_1}{d\sigma_2} = \vec{Q}''(\sigma_2), \beta = \frac{d\sigma_1}{d\sigma_2}.$$

For convex supporting curves $\beta > 0$, then line $\sigma_1 + \sigma_2 = Const$ intersects the kink line in a single point. For non-convex supporting curves $\beta + 1$ can change a sign. For the kink evolution some variants are possible (fig.8). Kinks appear and disappear lonely on the ends of string or by pairs inside it. Kinks in pair are oriented to the opposite sides at the moment of creating. (The direction of the kink is determined by vector $\vec{Q}_1'' \text{sgn}(\beta + 1)$. Function \vec{Q}_1'' is continuous and does not vanish in the vicinity of the point C. While passing through the point C the value $\beta + 1$ reverses its sign.) The process of pair creating looks similarly in any other reference frame, because the Lorentz transformation does not change time ordering along the isotropic kink line. The point C of the world sheet, where a pair appears, has a complicated structure. In this point $\beta = -1$ ($\lambda = 1$), the precision of expansion (5) is insufficient for the surface investigation near this point. Points of single appearance of kinks on the edges also have a complicated structure. In these points $\beta = -1$ as well.

Non-singular excitations are also evident on figures, which propagate on the world sheet near non-convex parts of the supporting curve. These excitations are only the mappings (1) of non-convex interval AB onto string. However, another interpretation is possible. One can say that the excitation reflects from the end of string, during the reflection the non-convex interval on the trajectory of string end arises. In this the excitation becomes singular – the kink arises on string. The kink is absorbed on the end of string and inflects its trajectory. Besides, the excitations on string can decay into kink pairs.

Our approach is specific because we recover the shape of the world sheet from the given edge. In the usual statement the initial data for Cauchy problem are specified: initial coordinates and momenta of all

parts of string. The supporting curve is defined by this data:

$$Q_\mu(\sigma) = x_\mu(\sigma) + \int_0^\sigma d\sigma' p_\mu(\sigma').$$

The both approaches are equivalent, but they differ in their interpretation.

Condition of parallelism of tangents conserves in shifts

$$\sigma_1 \rightarrow \sigma_1 + 2\pi n_1, \quad \sigma_2 \rightarrow \sigma_2 + 2\pi n_2.$$

Images of the kink line in these shifts are shown on fig.9. Kinks appear and disappear on one edge of string, then after a lapse of time, equal to a semi-period P_0 , these processes repeat themselves on the another end. These processes may be overlapped in time – the new kink can appear before the disappearance of the preceding one, a few kinks created by one and the same pair of inflections can be simultaneously placed on string. But the kink lines cannot have stable intersections.

Suppose, the kink lines intersect each other in a point (σ_1^*, σ_2^*) . This implies that the equation $\varphi(\sigma_1) = \varphi(\sigma_2)$ has two solutions in any vicinity of the point (σ_1^*, σ_2^*) . Let us expand $\varphi(\sigma)$:

$$\begin{aligned} \varphi(\sigma_1^*) + \varphi'(\sigma_1^*)(\sigma_1 - \sigma_1^*) + \frac{1}{2} \varphi''(\sigma_1^*)(\sigma_1 - \sigma_1^*)^2 + \dots = \\ = \varphi(\sigma_2^*) + \varphi'(\sigma_2^*)(\sigma_2 - \sigma_2^*) + \frac{1}{2} \varphi''(\sigma_2^*)(\sigma_2 - \sigma_2^*)^2 + \dots \\ \varphi(\sigma_1^*) = \varphi(\sigma_2^*) \end{aligned}$$

If any of the first derivatives here does not vanish, then the equation can be unambiguously solved for σ_1 or σ_2 . Hence, the case $\varphi'(\sigma_1^*) = \varphi'(\sigma_2^*) = 0$ is of our interest:

$$\sigma_2 - \sigma_2^* = \pm \sqrt{\frac{\varphi''(\sigma_1^*)}{\varphi''(\sigma_2^*)}} (\sigma_1 - \sigma_1^*).$$

The function $\varphi(\sigma)$ must have the extrema of the same sign in the points σ_1^*, σ_2^* . The inflections of the curve $\vec{Q}(\sigma)$ are placed in these points. Between the points σ_1^* and σ_2^* an extremum of the opposite sign is placed, so the total number of inflection points must be greater than two. The tangents in the inflection points σ_1^* and σ_2^* must be parallel to each other, the kink lines in the point (σ_1^*, σ_2^*) are also aligned in this direction. Therefore, a tangency of kink lines occurs in the case considered, see the central position in fig.10. This singularity is not stable. In small deformations of the supporting curve the tangents in the inflection points become non-parallel, then kink lines disconnect and have no common points. Pay attention to the fact that on the right position of fig.10

the point C appears, where the value $\beta + 1$ changes its sign. In this point the kinks annihilate, later a new kink pair appears in the point C'.

The kink line on the world sheet can be formed in a closed contour. This takes place, when the equation $\varphi(\sigma_1) = \varphi(\sigma_2)$ has its solutions in a vicinity of extrema with the opposite signs (fig.11). If one lifts the minimum and lowers the maximum, then the contour contracts and disappears in a moment when the tangents in the inflection points become parallel (when the minimum and the maximum are equal).

When $\nu \neq 1$, $\nu - 1$ kinks are constantly present on string. Inflection points do not significantly affect the motion of these kinks. An exception is the case, when in the interval between inflection points A and B the point σ_1 exists with the greater negative curvature $\frac{d\varphi}{d\sigma}$, then in the correspondent point σ_2 :

$$\varphi(\sigma_1) = \varphi(\sigma_2), \quad -\frac{d\varphi}{d\sigma}(\sigma_1) > \frac{d\varphi}{d\sigma}(\sigma_2) > 0.$$

In this case the points with a slope $1/\beta = \frac{\varphi'(\sigma_1)}{\varphi'(\sigma_2)} < -1$ are present on the kink line, time $x^0 \sim \sigma_1 + \sigma_2$ changes non-monotonously along the kink line (fig.12).

Notion about light cone gauge in $d = 3$

The following fact shows that LCG works poorly in $d = 3$. Smooth closed curves on a plane are subdivided into classes with respect to the winding number ν , which cannot be continuously deformed to each other. On the other hand, a shape of the world sheet in LCG is uniquely determined by function $Q_2(\sigma)$ (in space $d = 3$ vector \vec{Q}_\perp has only one component), these functions form a connected set.

Let's consider a point σ^* on the supporting curve $\vec{Q}(\sigma)$, in which tangent vector $\vec{Q}'(\sigma)$ is directed along the first axis (fig.13). In RFG for this point the following relation holds:

$$\frac{dQ_0}{dL} = \frac{dQ_1}{dL}, \quad \sigma = \sigma^*.$$

But in LCG

$$Q'_0 - Q'_1 = \frac{dL}{d\sigma} \left(\frac{dQ_0}{dL} - \frac{dQ_1}{dL} \right) = \frac{P_-}{\pi} = Const,$$

that's why $\frac{dL}{d\sigma} \rightarrow \infty$, when $\sigma \rightarrow \sigma^*$. In RFG $Q_0 \sim L$, then function $Q_0(\sigma)$ is non-smooth in the point $\sigma = \sigma^*$, function $\vec{Q}(\sigma)$ is also non-smooth:

$$\begin{aligned} \sigma \rightarrow \sigma^*, \quad Q'_1 &= Q'_0 - \frac{P_-}{\pi} \rightarrow \infty, \\ Q'_+ &= 2Q'_0 - \frac{P_-}{\pi} = \frac{\pi}{P_-} Q_2'^2 \rightarrow \infty, \quad Q'_2 \rightarrow \infty. \end{aligned}$$

Therefore, LCG parametrizes the supporting curve irregularly in the vicinity of σ^* . This complicates an analysis of curves smoothness in LCG parametrization.

The supporting curve has no points with tangent directed along \vec{e}_1 , only if $\nu = 0$. These curves have at least two inflection points. The motion of kinks for such world sheets is viewed on fig.14.

For the supporting curves with $\nu \neq 0$ LCG parametrization is not regular. Fourier expansion of non-smooth functions necessarily has an infinite number of harmonics. Therefore, most of string configurations are infinite-modal in LCG when $d = 3$. Apparently, this makes string quantization in LCG impossible. Note, that the common obstacle for string quantization – anomaly is absent in $d = 3$. Quantization in LCG brings the anomaly in commutation relations of rotation generators in space, transverse to momentum [8]. When $d = 3$ this space is two-dimensional, it has only one rotation generator.

4. Singularities on world sheets in $d = 4$ – pinch points

When $d = 4$, the space, orthogonal to momentum P_μ , is three-dimensional. In RFG vector $\vec{Q}'(\sigma)$ lies on a sphere. Singularities on the world sheet correspond to points of self-intersection of the curve $\vec{Q}'(\sigma)$. For supporting curves in general position singularities on world sheet are placed in isolated points (fig.15). (Singular points form kink lines in a special case, if the curve $\vec{Q}'(\sigma)$ is multiple.)

The shape of the world sheet in the vicinity of the singular point is determined by expression (4). In this vectors Q'_1, Q'_1'' and Q'_2 can be linearly independent, because in $d = 4$ a space, orthogonal to Q'_1 , containing these vectors, is three-dimensional.

$$\begin{aligned} x(\sigma_1 + \Delta\sigma_1, \sigma_2 + \Delta\sigma_2) = & \frac{1}{2} (Q_1 + Q_2) + \\ & + \frac{1}{2} Q'_1 \xi + \frac{1}{8\lambda^2} (\lambda^2 Q'_1'' - Q'_2'') \xi \eta + \\ & + \frac{1}{16\lambda^2} (\lambda^2 Q'_1'' + Q'_2'') (\xi^2 + \eta^2) + o(\xi^2 + \eta^2), \end{aligned} \quad (6)$$

$$Q'_2 = \lambda Q'_1, \quad \xi = \Delta\sigma_1 + \lambda\Delta\sigma_2, \quad \eta = \Delta\sigma_1 - \lambda\Delta\sigma_2.$$

The surface $(X, Y, Z) = (\xi, \xi\eta, \xi^2 + \eta^2)$ is displayed on fig.16. The projection of world sheet near the isolated singular point into any 3-dimensional space looks similarly (fig.17). Such singular points are called pinch points. In the pinch point the world sheet has no tangent plane. As the pinch point is approached along different paths, tangent planes come to different limiting positions. All these positions lie in 3-dimensional isotropic space, determined by vectors Q'_1, Q'_1'', Q'_2 .

At the moment of passing through the pinch point string undergoes an instantaneous cusp.

In RFG $Q_1''^0 = Q_2''^0 = 0$, $\lambda = 1$, therefore on the slice of the surface (6) by the plane $x^0 = Const$ near the pinch point the equation $\xi = Const$ holds. In general position vectors $\vec{Q}_1'' - \vec{Q}_2''$ and $\vec{Q}_1'' + \vec{Q}_2''$ are linearly independent. In the coordinate frame (Y, Z) , composed by these vectors, string has a shape of parabola

$$(\xi\eta, \xi^2 + \eta^2) = (Y, \xi^2 + Y^2/\xi^2).$$

When the pinch point is approached $\xi \rightarrow 0$, the parabola degenerates into a ray along the positive direction of Z axis. At this moment string has a cusp.

Many interesting properties of pinch points are described in the remarkable book of G.K.Francis [9]. We note here one property: in any projection on figure plane the pinch point lies on a contour of the surface, visible or hidden by other parts of the surface.

In the pinch point the lines of self-intersection of the surface terminate. Note, that the lines of self-intersection are present only on the projection of the world sheet in 3-dimensional space. The world sheet shown in fig.17 has no self-intersections in 4-dimensional space, because any string forming this surface does not intersect itself. Stable self-intersections of the world sheet in 4-dimensional space can be located in isolated points, as it is shown in fig.18.

A relation between singularities on sheets in $d = 3$ and in $d = 4$ is demonstrated by the following puzzle. Curves with different winding numbers ν cannot be transformed to each other by the continuous deformation on a plane. They can be transformed to each other by “twisting through the third dimension” (fig.19). What will happen with the world sheet in such deformation? The answer is given in fig.20.

As soon as the supporting curve $\vec{Q}(\sigma)$ becomes non-flat, the kink line is smoothed down. The projection of the world sheet into rest frame intersects itself along the line PQ , beginning in pinch point Q and terminating on the edge of the sheet. In further deformation the pinch point moves to the edge of the sheet and disappears there. Then the supporting curve becomes flat with $\nu = 1$, the world sheet has no singularities.

In the inverse order: the sheet turns up on the edge, the double line and the pinch point appear. Then the surface is rumpled along kink line.

Small deformations of the supporting curve into the third dimension can create a number of pinch points. In described deformation the pinch points disappear in pairs, as is shown in fig.21 (compare with fig.9,10 in [9]).

Notion about light cone gauge in $d = 4$

RFG and LCG parametrizations are connected by stereographic projection (fig.22):

$$\vec{Q}'_{\perp}(LCG) = \left(\frac{d\vec{Q}}{dL} \right)_{\perp} \frac{dL}{d\sigma} = \left(\frac{d\vec{Q}}{dL} \right)_{\perp} \frac{P_-/\pi}{1 - \frac{dQ_1}{dL}},$$

$$\frac{BD}{CE} = \frac{AB}{AE},$$

$$BD = |\vec{Q}'_{\perp}(LCG)|, \quad CE = \left| \left(\frac{d\vec{Q}}{dL} \right)_{\perp} \right|,$$

$$AO = 1, \quad AE = 1 - \frac{dQ_1}{dL}, \quad AB = \frac{\sqrt{P^2}}{\pi},$$

$P_- = P_0 = \sqrt{P^2}$ for $n_{\mu} = (1, 1, 0, 0)$ in the reference frame, connected to momentum $P_{\mu} = (\sqrt{P^2}, 0, 0, 0)$.

When $d = 4$, supporting curves also exist which have a tangent $\vec{Q}'^{(RFG)}(\sigma)$, directed along \vec{e}_1 in some point. Alignment of $\vec{Q}'(\sigma^*)$ along \vec{e}_1 can be removed by small deformation of the curve. Positions $\vec{Q}'(\sigma^*) \uparrow \vec{e}_1$ are not topological obstacles for deformations, their presence does not lead to the division of curves into non-equivalent classes. Smooth curves $\vec{Q}(\sigma)$ in RFG correspond to continuous curves $\vec{Q}'(\sigma)$ on a sphere. Smoothness of curve $\vec{Q}'(\sigma)$ can be violated (fig.23). Non-smoothness of the curve $\vec{Q}'(\sigma)$ does not necessarily mean the non-smoothness of correspondent functions in LCG and infinite number of modes in their Fourier expansions: not more than $\vec{Q}'' = 0$ is in this point.

All continuous curves on a sphere can be transformed to each other by continuous deformation. These deformations can be developed on a sphere with the rejected point \vec{e}_1 . This can require large deformations. Initial and final positions of the curve on fig.24 are close in RFG: $|\vec{Q}'(\sigma)_i - \vec{Q}'(\sigma)_f| < \varepsilon$, but in LCG they are not close (in the same sense), see fig.25.

5. Characteristic lines and causal structure of the world sheet

When the supporting curve $Q_{\mu}(\sigma)$ is non-smooth in a point σ_0 , the world sheet has angular points along a line

$$x_{\mu}(\sigma_0, \sigma) = \frac{1}{2} (Q_{\mu}(\sigma_0) + Q_{\mu}(\sigma)). \quad (7)$$

This line is proportional to the supporting curve with coefficient 1/2 (fig.26). Tangent planes to the world sheet on the fracture line are determined by vectors

$(Q'_\mu(\sigma_0-o), Q'_\mu(\sigma))$ and $(Q'_\mu(\sigma_0+o), Q'_\mu(\sigma))$, these planes do not coincide. Inside the regions of the world sheet, separated by line (7), condition (2) holds. On line (7) the conservation law holds: a momentum flowing from one part of the sheet is equal to that flowing into another part and is equal to

$$\frac{1}{2} \int_{\text{along line (7)}} Q'(\sigma) d\sigma.$$

Therefore, this piecewise smooth surface is minimal.

If higher derivative $Q^{(n)}$ is subjected to discontinuity in the point σ_0 , then the world sheet has discontinuity in derivatives of the same order on the line (7). If a parametric invariant of the curve $Q(\sigma)$ is available, which depends on derivatives up to $Q^{(n)}$ and undergoes discontinuity in the point σ_0 (i.e. not only parametrization of the curve is non-regular in the point σ_0 , but the curve itself is non-regular), then the world sheet has a parametric invariant depending on derivatives of the same order and undergoing discontinuity along line (7). For example, if the curvature of curve $\tilde{Q}(\sigma)$ has discontinuity in point σ_0 , then invariants of the second quadratic form of world sheet have discontinuity on line (7).

Therefore, irregularities of the supporting curve propagate on the world sheet along isotropic lines (7). Parameters (σ_1, σ_2) form isotropic coordinates on the surface: $(\partial_1 x)^2 = (\partial_2 x)^2 = 0$. One can easily show that equation (2) in these coordinates takes a form $\partial_1 \partial_2 x = 0$. The lines (7) are characteristics of equation. This is in agreement with a concept of *genes*, introduced in work [1], which implies that the irregularities of initial data propagate on the world sheet along characteristics.

Characteristics, edges and kink lines exhaust all isotropic lines on the world sheet:

$$(Q'_1 \dot{\sigma}_1 + Q'_2 \dot{\sigma}_2)^2 = 0 \Rightarrow (Q'_1 Q'_2) \dot{\sigma}_1 \dot{\sigma}_2 = 0,$$

- $\dot{\sigma}_1 \dot{\sigma}_2 = 0$:
 $\sigma_1 = Const$ or $\sigma_2 = Const$ characteristics
(through each regular point of the surface two characteristics pass);
- $(Q'_1 Q'_2) = 0$:
 $\sigma_2 = \sigma_1$ or $\sigma_2 = \sigma_1 + 2\pi$ the edges.
In other cases $\vec{Q}'_1 \vec{Q}'_2 = |\vec{Q}'_1| |\vec{Q}'_2| \Rightarrow Q'_1 \parallel Q'_2$ the kink line.

Characteristic marks the path of a light signal along the world sheet. Two characteristics pass through each inner point of the world sheet. Single characteristic passes through a point on the edge. Characteristics that pass through kink lines and edges are tangent to these curves (fig.27) (they are tangent in Minkowski space, but not on the parameters plane).

Causal structure

Characteristics introduce a causal structure on the world sheet. This structure determines, whether two points on the world sheet communicate by a signal, propagating along world sheet with a velocity, not greater than the light velocity. Characteristics, passing through a given point, divide the world sheet into a future region, a past region and regions of causal independence (fig.28). The singularities do not affect causal structure (the kink line is an obstacle for signals, the velocity of which is strictly less than the velocity of light, but is transparent for light signals).

Points of self-intersections make the causal structure to be more complex (the right part of fig.28). The point of self-intersection is marked by two separate points S and S' on the parameters plane, which are glued in Minkowski space. If the self-intersection point S is placed in the future region for point A , then the consequences of A are also S' , the future region for S' , the points of self-intersections, contained in F' etc.

Position of singular world sheets among other surfaces

The world sheet with singularities is not a special case of the world sheet, but on the contrary, it is a world sheet “in general position”. This position is quite analogous to the position of smooth non-monotonic function in a set of all smooth functions. A set of singular world sheets, endowed by the appropriate definition of world sheets closeness¹, forms a region. For the sheets inside the region the singularities cannot be eliminated by small deformation. Sheets without singularities lie on the boundary of the region. This means that in any vicinity of a sheet without singularities the singular world sheets are available, i.e. singularities are created by small deformation.

For general surfaces the singular points considered are not stable. The folds of surfaces in 3D space are eliminated by smoothing. For example, a half-cubic fold is eliminated by homotopy $(\xi, \eta^2, \eta^3) \rightarrow (\xi, \eta^2, \eta^3 + \epsilon\eta)$. The pinch points are stable in 3D space, but are not stable in 4D. The surface (6) lies in 3-dimensional subspace of 4-dimensional space (up to $o(\xi^2 + \eta^2)$). Homotopy $(\xi, \xi\eta, \xi^2 + \eta^2, 0) \rightarrow (\xi, \xi\eta, \xi^2 + \eta^2, \epsilon\eta)$ eliminates the singular point on this surface (for all $\epsilon \neq 0$ the tangent vectors to this surface are linearly independent).

Therefore, kink lines in 3D and pinch points in 4D are stable on minimal surfaces, but can be eliminated by deformation, violating the minimal property. These singular points are specific namely for minimal surfaces. The singularities induced by non-smoothness of the supporting curve are not stable (they can be eliminated by smoothing). The lines of self-intersection in 3D and points of self-intersections in 4D are stable for all surfaces.

There are no stable singular points on the world sheets in the space with $d > 4$, because in this case the curve $\vec{Q}'(\sigma)$ lies on $(d - 2)$ -dimensional sphere,

¹For example, uniform closeness of derivatives $\vec{Q}'(\sigma)$ in RFG, see the end of Section 4.

and its self-intersections can be eliminated by small deformations.

6. Outlook

We have studied the singularities on the world sheets of open string.

When $d = 3$, singularities have the form of kinks, propagating on string with the light velocity. If projection of the supporting curve into the rest frame is the convex curve with winding number ν , then the number of kinks conserves and equals $\nu - 1$. If this curve is non-convex, then new kinks periodically appear and disappear on string. Kinks appear and disappear lonely on the ends of string or in pairs inside it.

When $d = 4$, the singularities on the world sheet are placed in isolated pinch points. The string undergoes an instantaneous cusp at the moment of passing through the pinch point.

When $d > 4$, there are no stable singularities on the world sheets.

If the supporting curve itself has singular points, then the correspondent singularities on the world sheet are placed on characteristics, passing through these points.

We will study the singularities on the world sheets of other topological types and singularities in processes of string breaking elsewhere.

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References

- [1] Artru X. Classical String Phenomenology – Preprints LPTHE 78/25 (1978), 79/8 (1979), 81/1 (1981). /Laboratoire de Physique Théorique et Hautes Energies, Université de Paris-Sua, 91405, Orsay, France/.
- [2] Pron'ko G.P. *et al.* // Elementary Particles and Atomic Nuclei Journal 14 (1983) 3, 558 (russian).
- [3] Georgadze G.P. *et al.* // Theor. Math. Phys. 40 (1979) 221 (russian); Pogrebkov A.K. // Theor. Math. Phys. 45 (1980) 161 (russian).
- [4] Pron'ko G.P. private communication.
- [5] Pron'ko G.P. // Rev. Math. Phys. 2,3 (1990) 355;
Pron'ko G.P. // Theor. Math. Phys. 59,2 (1984) 240 (russian).

- [6] Nikitin I.N., Talanov V.V. On Drawing Complicated 3D Surfaces – Preprint IHEP 94-23, 1994.
Klimenko S.V., Nikitin I.N., Talanov V.V., Report on 2nd Dagstuhl seminar on scientific visualization, Dagstuhl, Germany, 23-27 May 1994.
- [7] Zheltuhin A.A. // Nuclear Physics Journal. 34,2 (1981) 562 (russian).
- [8] Nikitin I.N. // Nuclear Physics Journal. 56,9 (1993) 230 (russian).
- [9] Francis G.K., A Topological Picturebook, Springer-Verlag 1987,1988, (Moscow, Mir, 1991, russian).

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Captions to figures

Fig. 1 World sheet of string is constructed as geometrical place of middles of segments, connecting all possible pairs of points on the supporting curve. It has two isotropic edges: the supporting curve (1) and its image in translation onto the semi-period (2). A slice of the world sheet by the plane of constant time gives string in this instant of time. Time axis is directed from left to right.

Fig. 2 A point on a surface is regular, if tangent vectors are linearly independent in it. Otherwise the point is singular. The numbers on this figure denote: 1 – regular point, 2,3 – singular points. The point 2 lies on the edge of the world sheet.

Fig. 3 The shape of the world sheet near singular line.

Fig. 4 Kink on string (in the rest frame). The numbers on the figure denote: 1 – supporting curve, 2 – trajectory of the kink, 3 – the string. Direction of the kink \vec{Q}'' , direction of its velocity \vec{Q}' and direction of motion of the string ends are related (see text).

Fig. 5 Left upper part shows the world sheet for the supporting curve, shown on the previous figure. Left bottom part shows the projection of this world sheet into the rest frame. Numbers denote: 1,2 – the edges of the sheet, 3 – kink line. Note, that near a point Q the edge 1 passes at first in front of the kink line, then behind it. Near the point Q the world sheet has self-intersection. Right part displays (from top to bottom): the kink, the line of self-intersection PQ , the self-intersection of string.

Fig. 6 Non-convex supporting curve and its unit tangent vector. A,B – inflection points. The tangents in 1,2,3 are collinear, therefore additional kink lines appear on string.

Fig. 7 Polar angle of tangent vector dependence of σ and the kink line on a plane of parameters (σ_1, σ_2) . a, b, c – the intervals of monotonicity of the function $\varphi(\sigma)$. 1,2,3 – the points with parallel tangents, see previous figure.

Fig. 8 The world sheets for non-convex supporting curves. Left upper parts show the kink line on the plane of parameters (σ_1, σ_2) .

- a) Kink appears on the end of string at the moment, when the end passes through the inflection point A. Kink disappears on the same end when it passes through the inflection point B. Between the points A and B the world sheet intersects itself along the line PQ .
- b) At some moment in the point C two kinks appear on string. One kink disappears on the end of string in the point A, another one disappears in the point B.
- c) In the point C the pair of kinks appears. One kink disappears on the end of string in the point A. In the point B new kink appears, it annihilates the second kink in the point C'.

Fig. 9 Images of the kink line in translations $\sigma_1 \rightarrow \sigma_1 + 2\pi n_1$, $\sigma_2 \rightarrow \sigma_2 + 2\pi n_2$, which do not change the parallelism of tangents. The kinks appear and disappear on one end of string, then after a lapse of time equal to the semi-period P_0 these processes are repeated on another end.

Fig. 10 Rearrangement of kink lines. In deformation of the supporting curve the kink lines approach each other, become tangent, then the scattering of the kinks transfers to “the annihilation channel”.

Fig. 11 Closed kink line on the world sheet. The kink pair appears in the point C and disappears in the point C'. The supporting curve has 4 inflection points.

Fig. 12 The world sheet for the non-convex supporting curve with $\nu = 2$. One kink (a) is constantly present on string. At some moment in the point C the kink pair appears, one kink of which annihilates kink (a) in the point C'. Another kink continues to move on string.

Fig. 13 In the point σ^* tangent vector to the supporting curve is directed along axis 1. Function $Q_0(\sigma)$ has an infinite derivative in this point.

Fig. 14 Upper figure. Supporting curve with $\nu = 0$. Its unit tangent vector has two cusps A and B, correspondent to the inflection points on the supporting curve. Polar angle $\varphi(\sigma)$.

Bottom figure. The world sheet and a shape of string. Solutions of equation $\varphi(\sigma_1) = \varphi(\sigma_2)$ on the plane (σ_1, σ_2) . Their images in shifts $\sigma_1 \rightarrow \sigma_1 + 2\pi n_1$, $\sigma_2 \rightarrow \sigma_2 + 2\pi n_2$. The kink appears at one end of string and disappears on another end.

Fig. 15 The world sheet in $d = 4$ has singularities, when a curve $\vec{Q}'(\sigma)$ has points of self-intersection.

Fig. 16 The world sheet in the vicinity of isolated singular point. Axes form a basis in some 3-dimensional isotropic space (tangent to light cone).

Fig. 17 Projection of the world sheet into the rest frame near pinch point Q . This surface intersects itself along the curve PQ , terminating in the pinch point. String has a cusp in the pinch point.

Fig. 18 Self-intersection of string.

Fig. 19 Smooth closed curves with different winding numbers are not topologically equivalent on the plane, but are equivalent in 3D space.

Fig. 20 The deformation of the world sheet, correspondent to the deformation of the supporting curve on previous figure. Near each world sheet the curve $\vec{Q}'(\sigma)$ is also displayed.

Fig. 21 Annihilation of the pinch points.

Fig. 22 RFG and LCG parametrizations are related by stereographic projection.

Fig. 23 The curves $\vec{Q}'(\sigma)$ are continuous. Smoothness of $\vec{Q}'(\sigma)$ can be violated in the deformation of smooth curves $\vec{Q}(\sigma)$. This happens when the pinch point disappears on the edge, see fig.20.

Fig. 24 Two close curves i and f can be transformed one to another by deformation on a sphere with rejected point.

Fig. 25 The deformation, displayed on previous figure, on the plane \vec{Q}'_{\perp} in LCG.

Fig. 26 Fracture of the world sheet, caused by non-smoothness of the supporting curve. Line of fracture is obtained from the supporting curve by contraction in 2 times to point σ_0 .

Fig. 27 Characteristics are tangent to the edges and the kink line. Grid on the world sheet is composed of the characteristics. All characteristics are proportional to the supporting curve with the coefficient $1/2$ and therefore are congruent. The edges 1,2 and the kink line k are marked. The kink line divides the world sheet into two parts. The marked characteristic passes from one part to another through the kink line and remains smooth. Time required for propagation of the characteristics from one edge to another is constant and equals P_0 .

Fig. 28 Causal structure on the world sheet. F – future region, P – past region, I – causally independent regions.

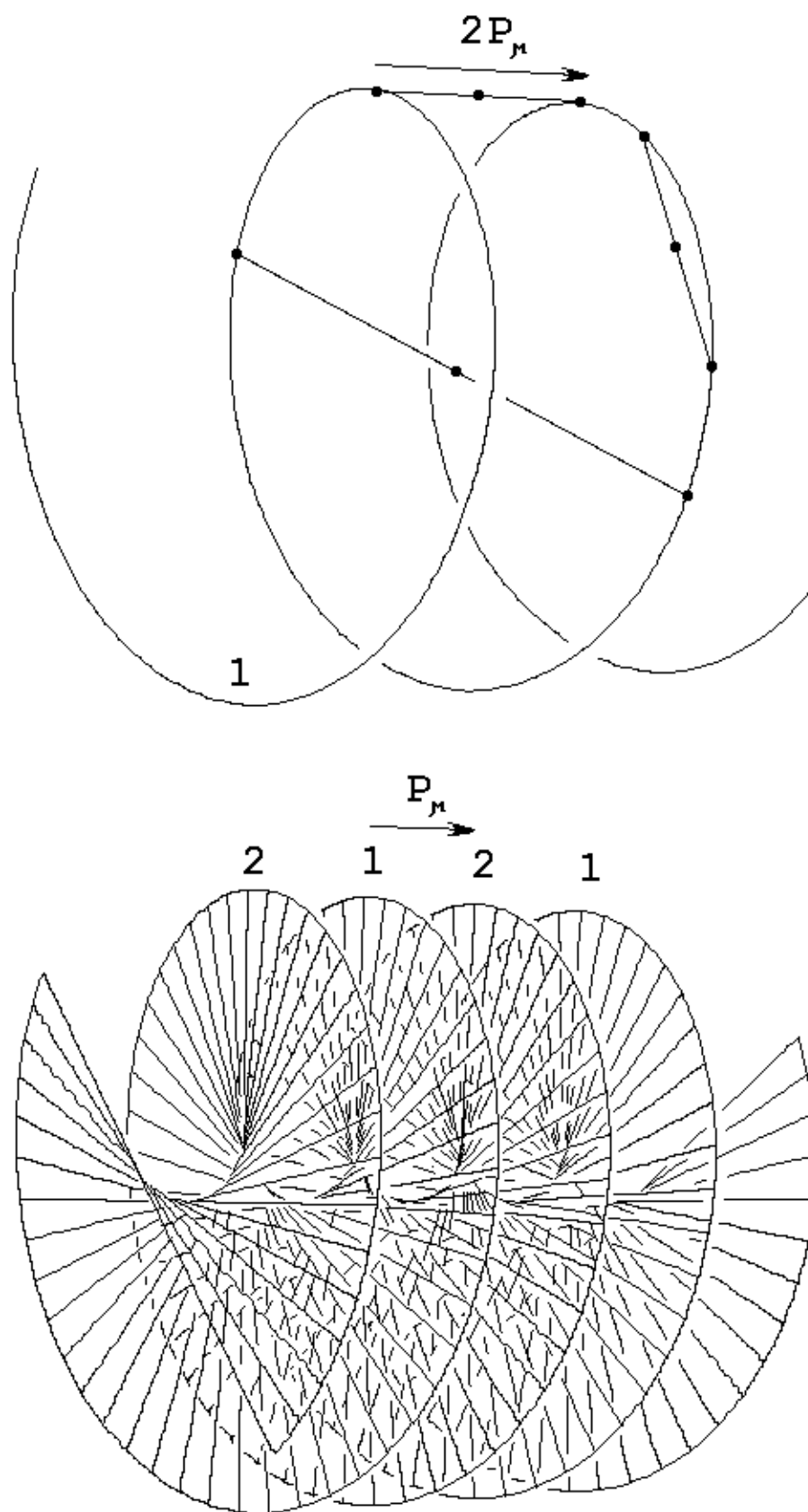


fig.1

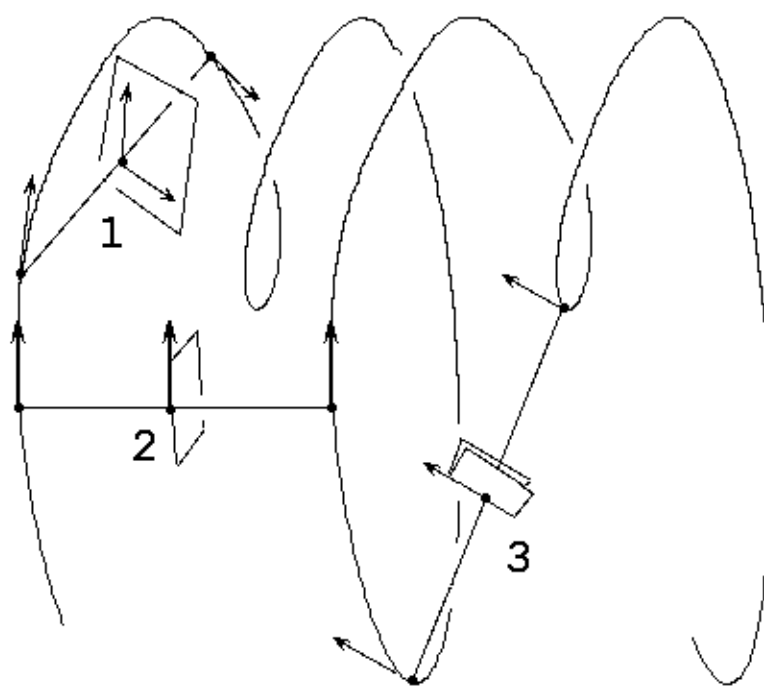


fig. 2

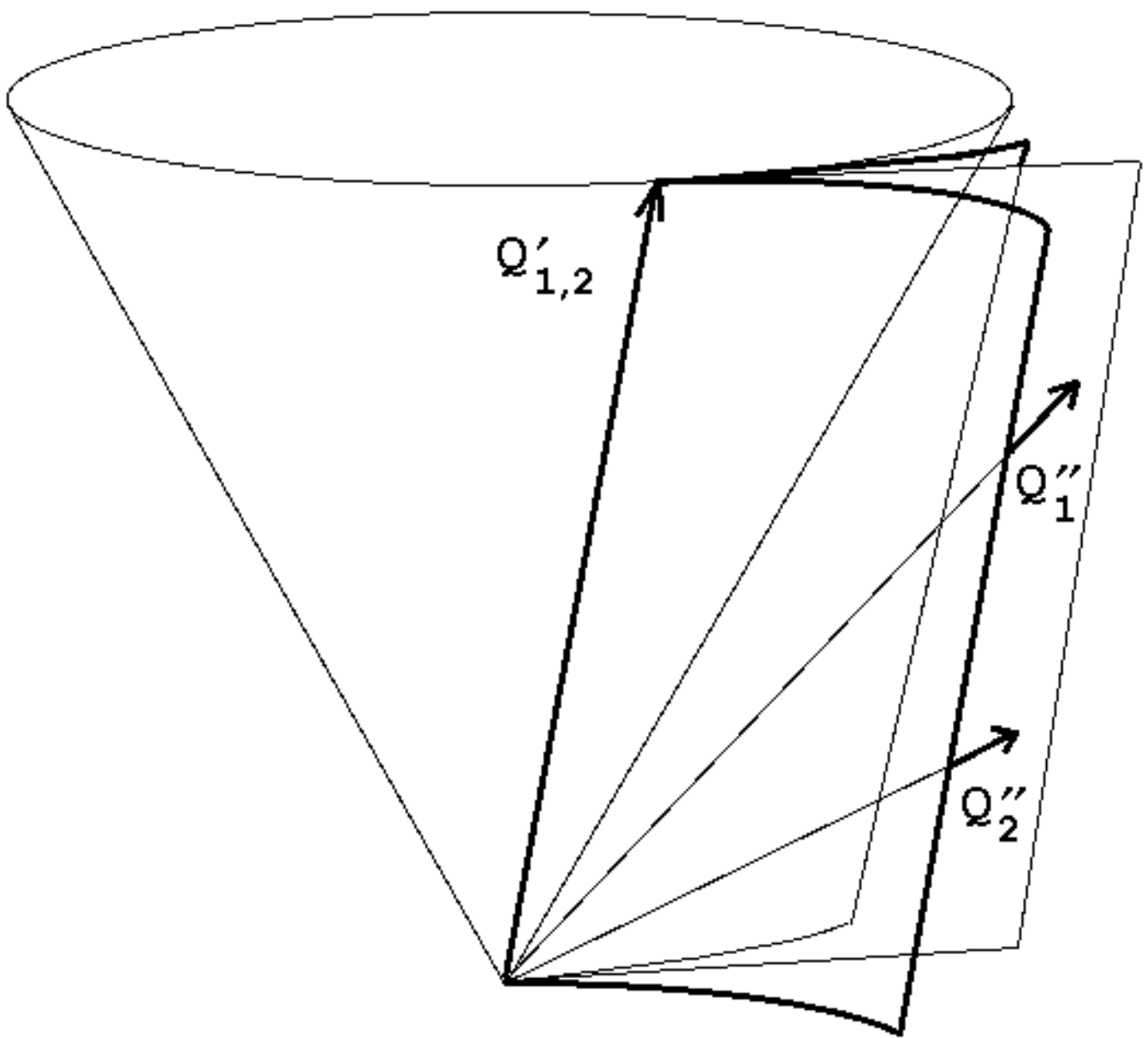


fig. 3

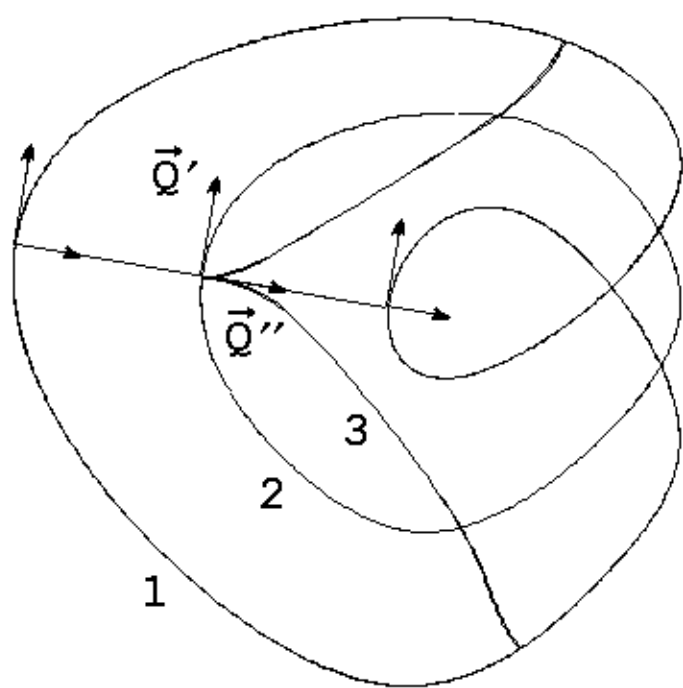


fig. 4

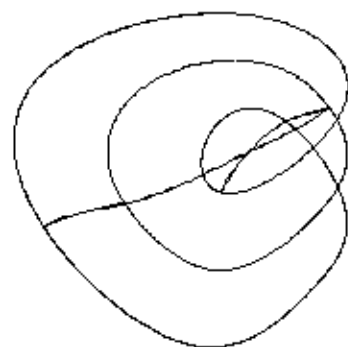
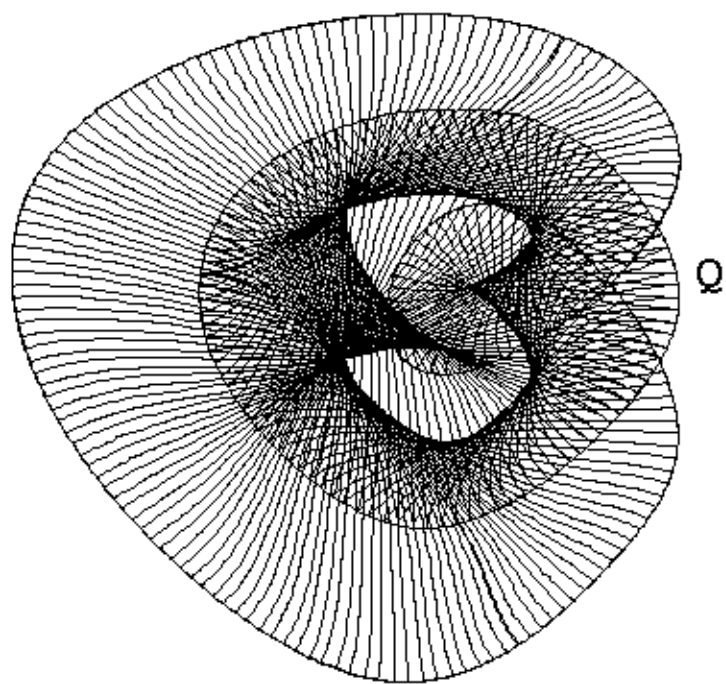
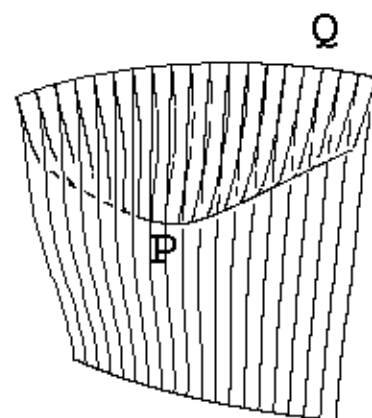
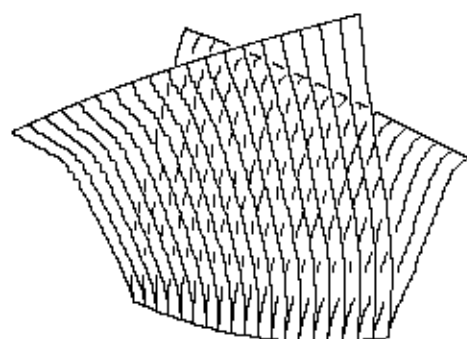
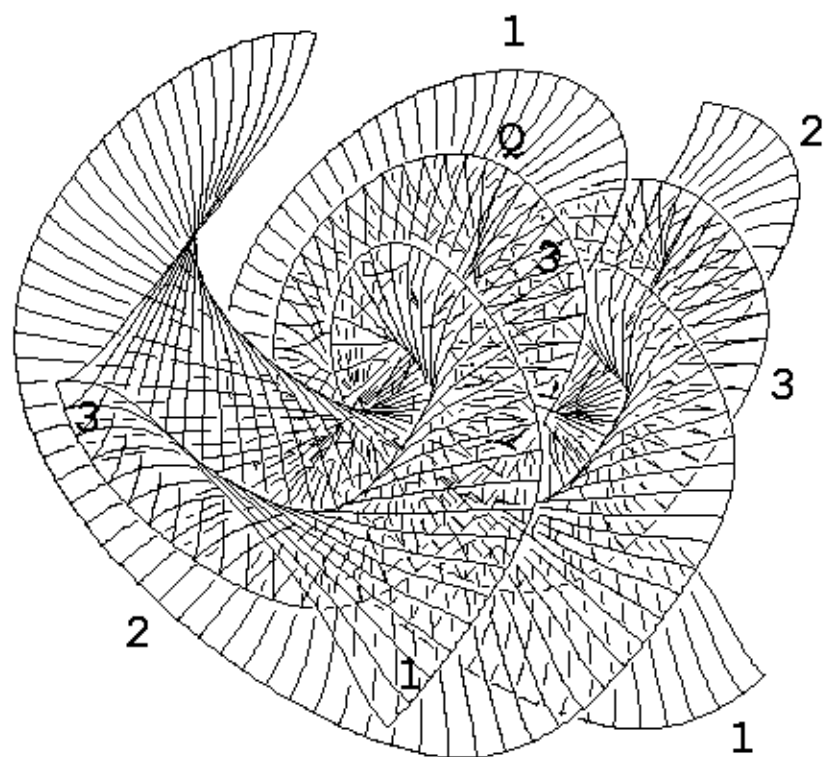


fig. 5

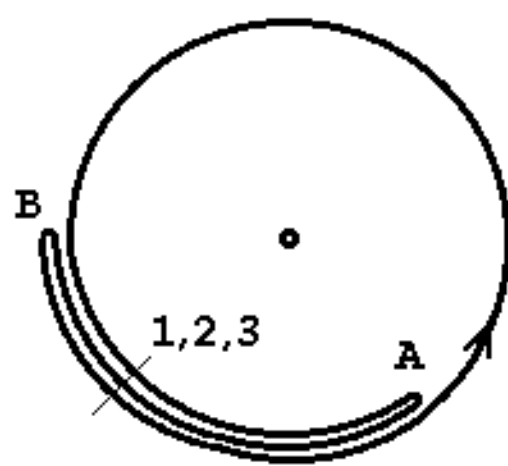
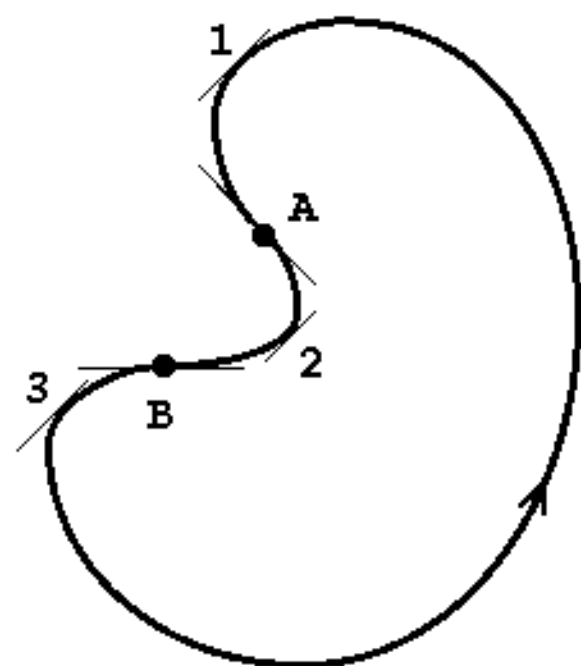


fig. 6

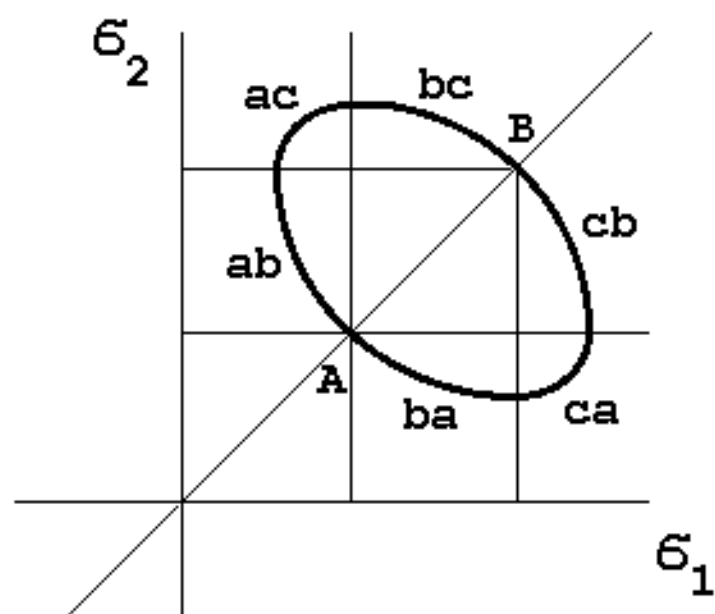
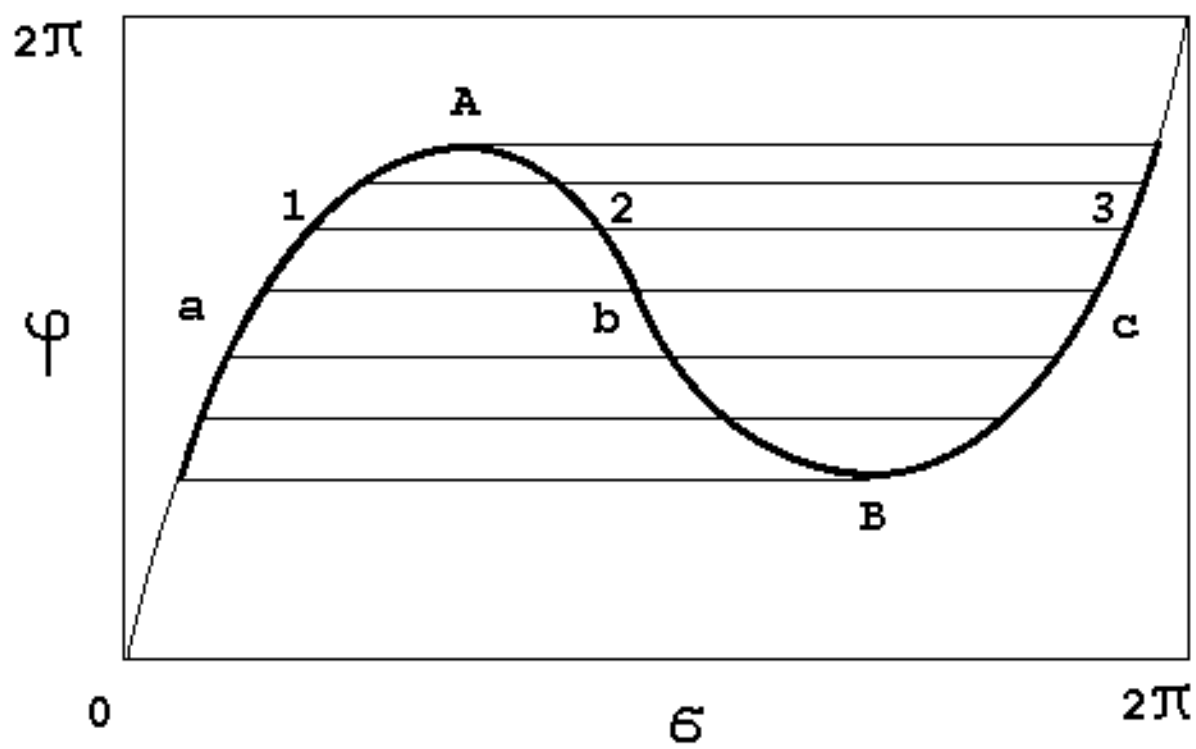


fig.7

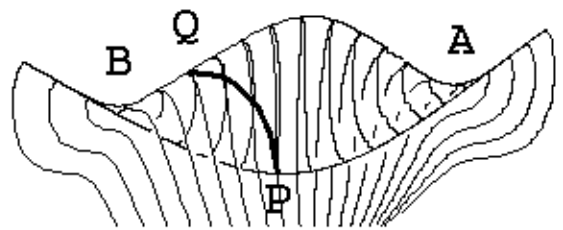
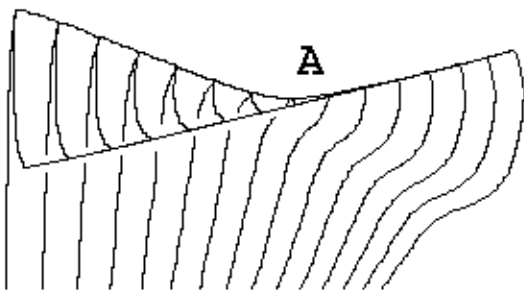
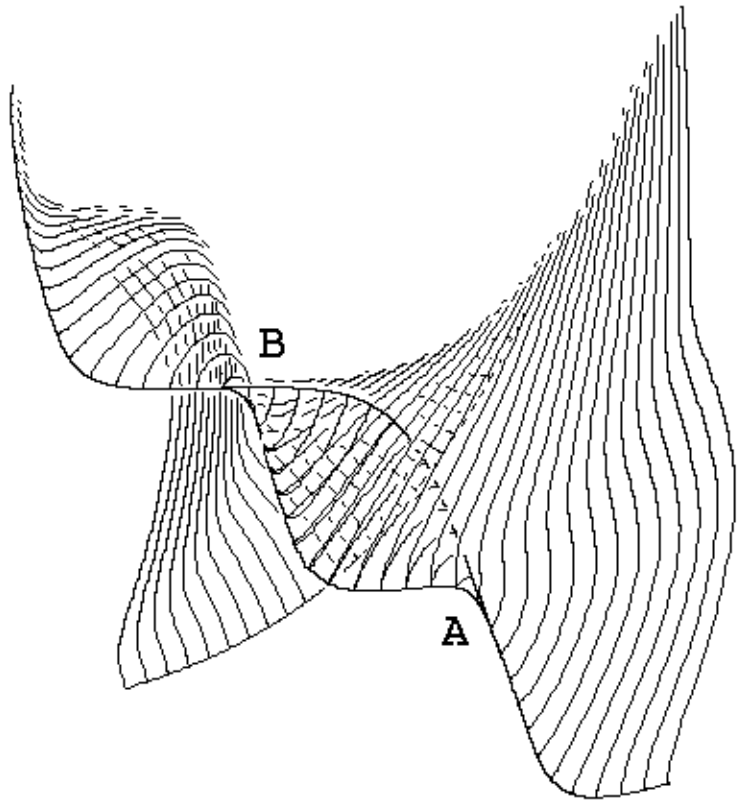
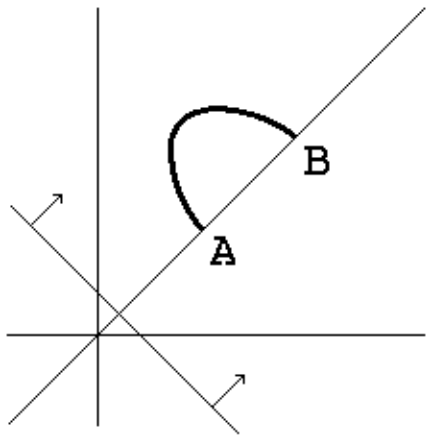


fig. 8a

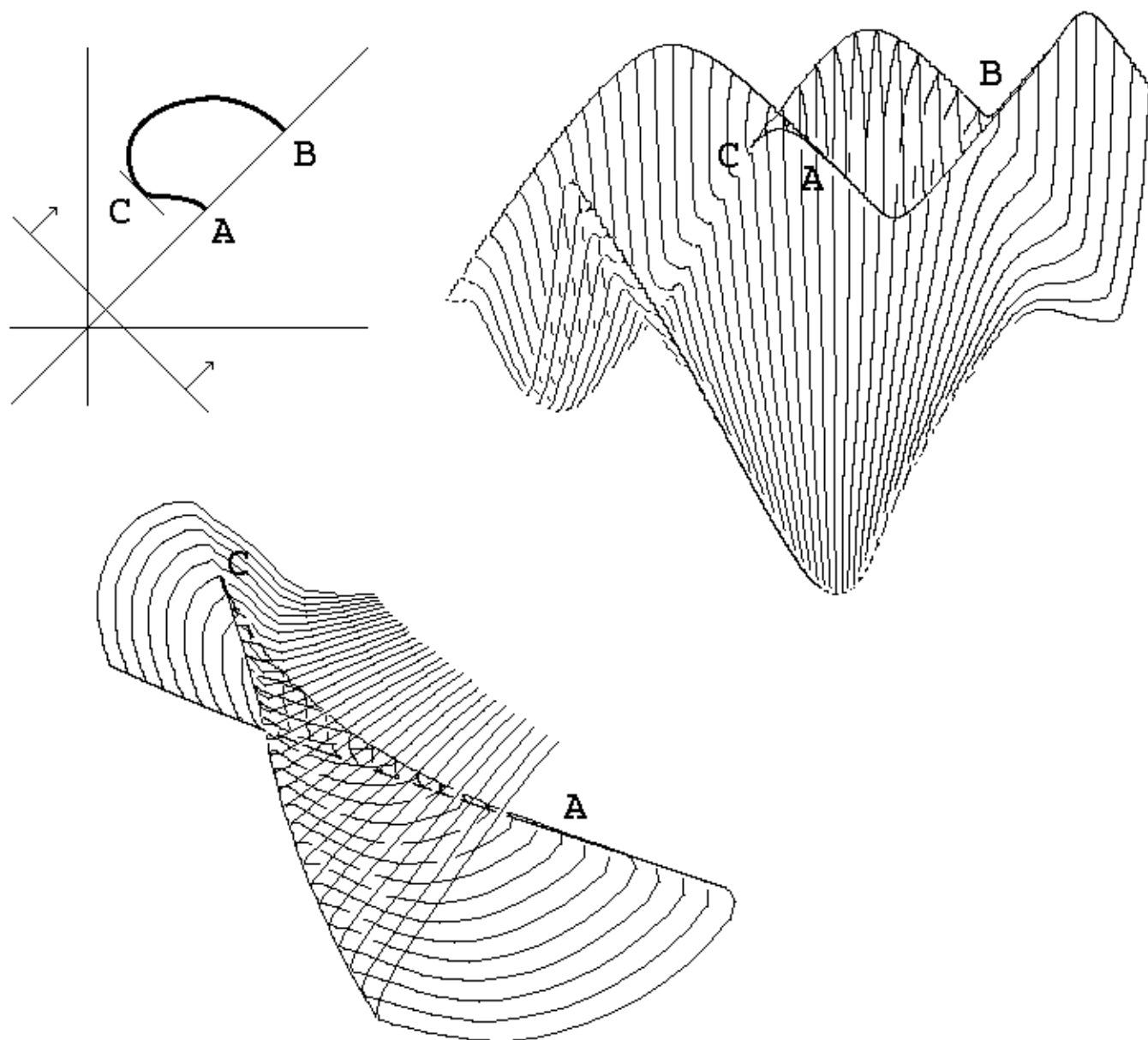


fig. 8b

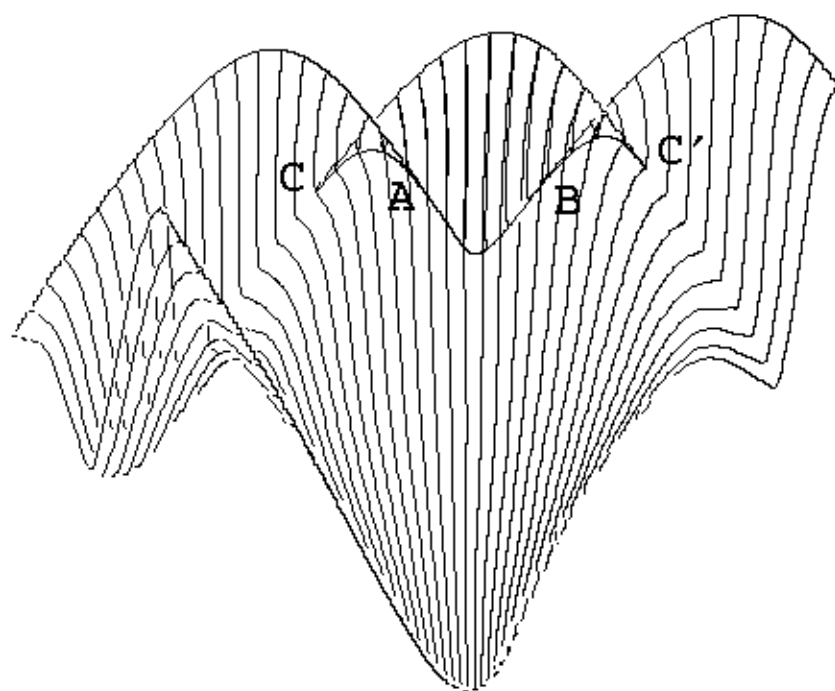
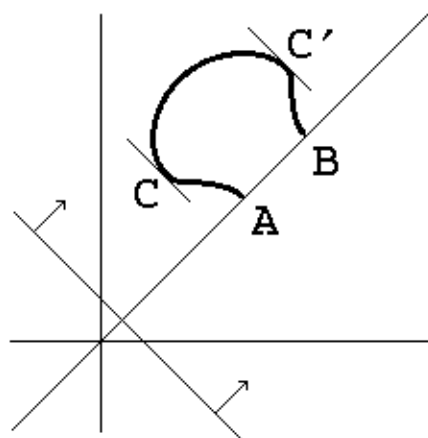


fig. 8c

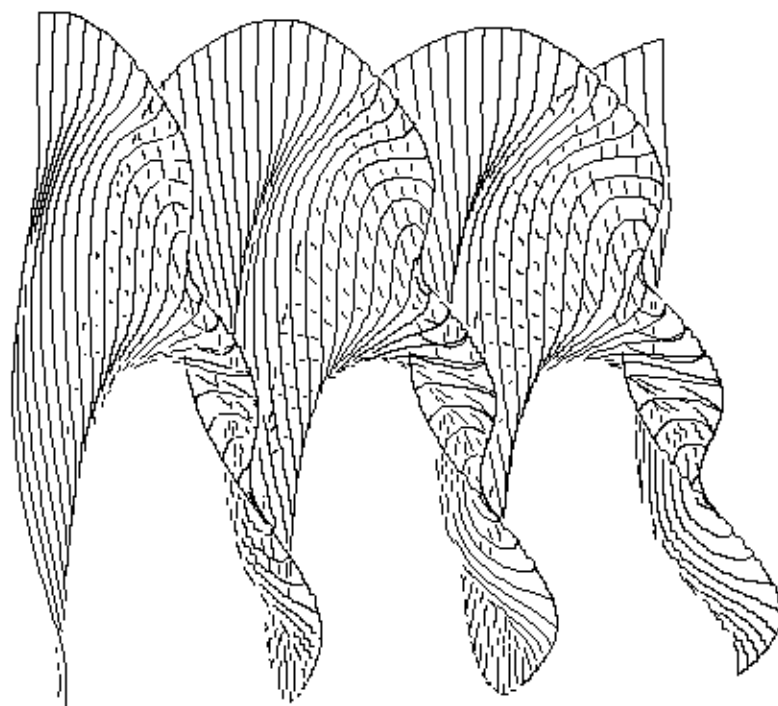
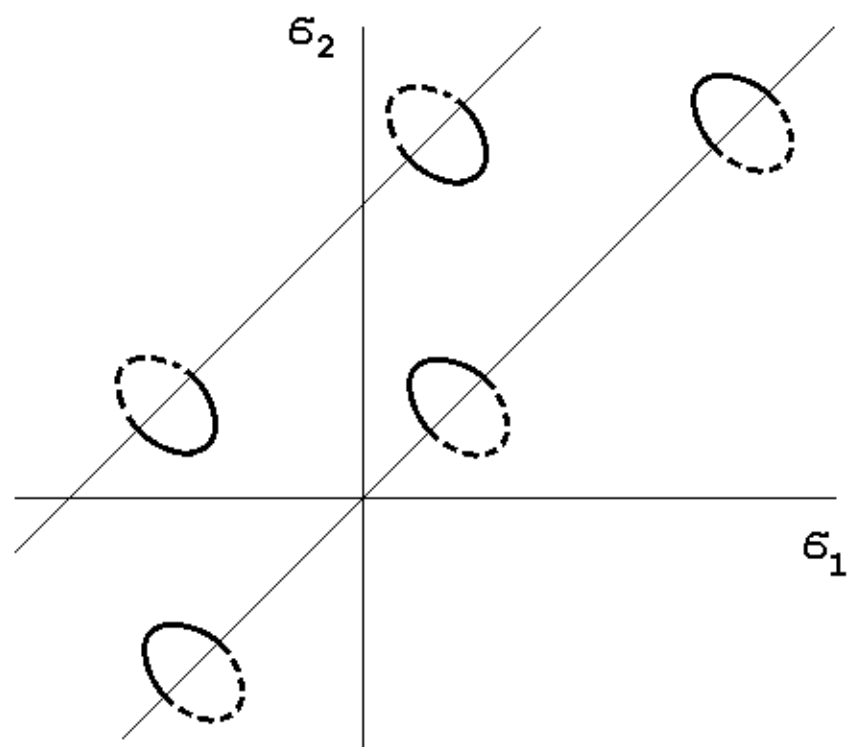


fig. 9

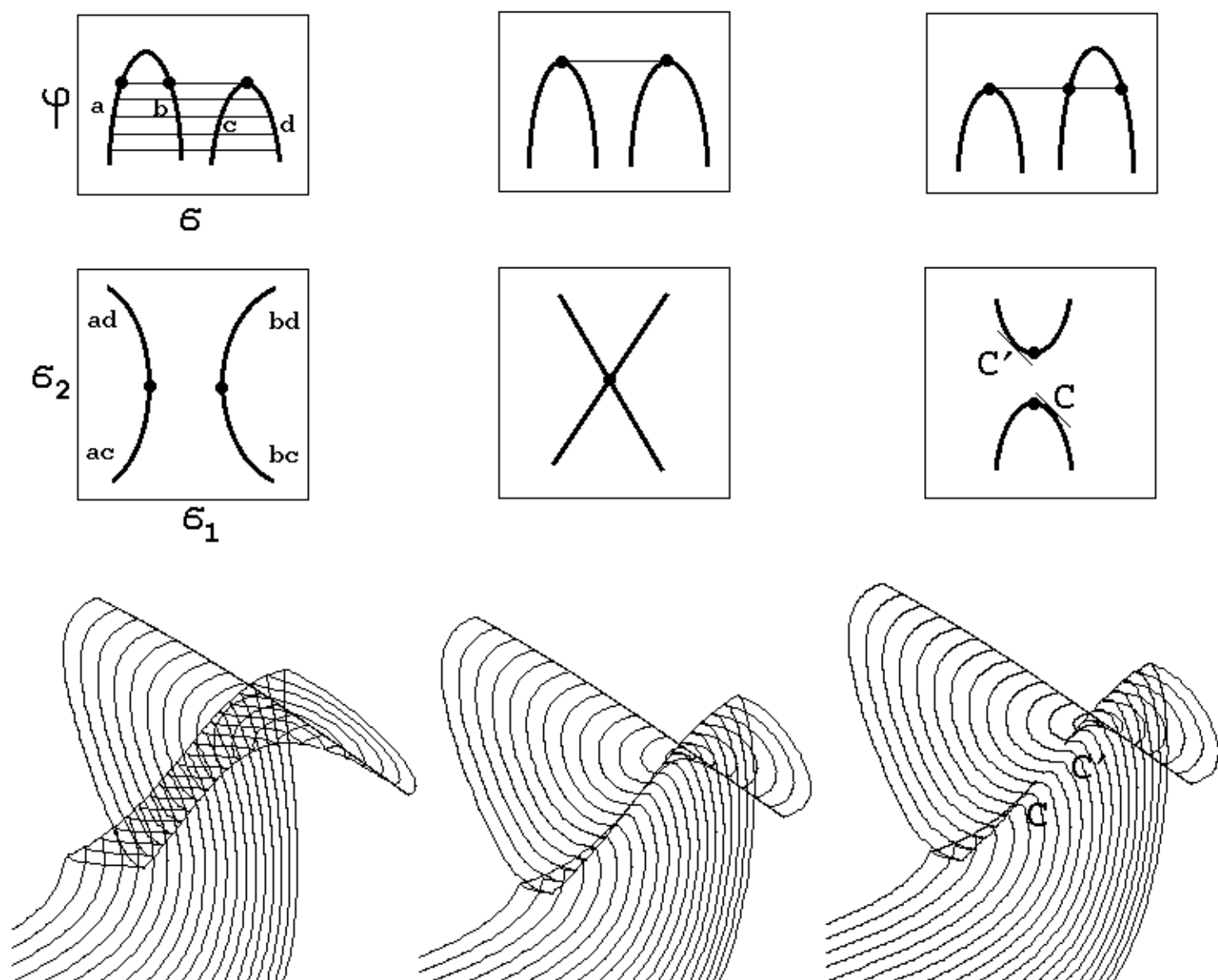


fig.10

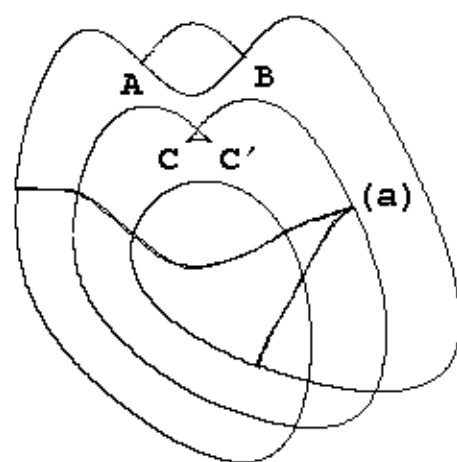
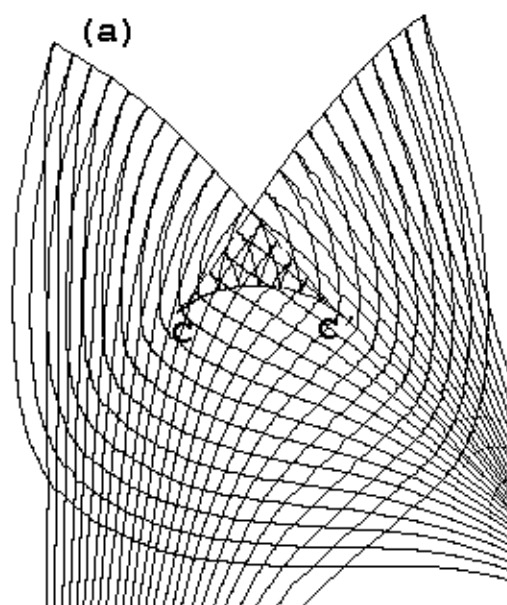
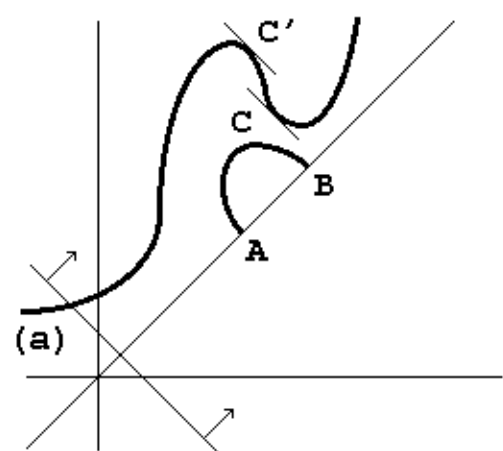
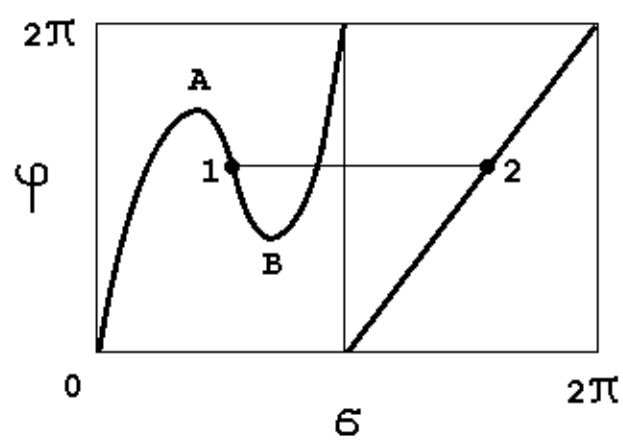


fig.11

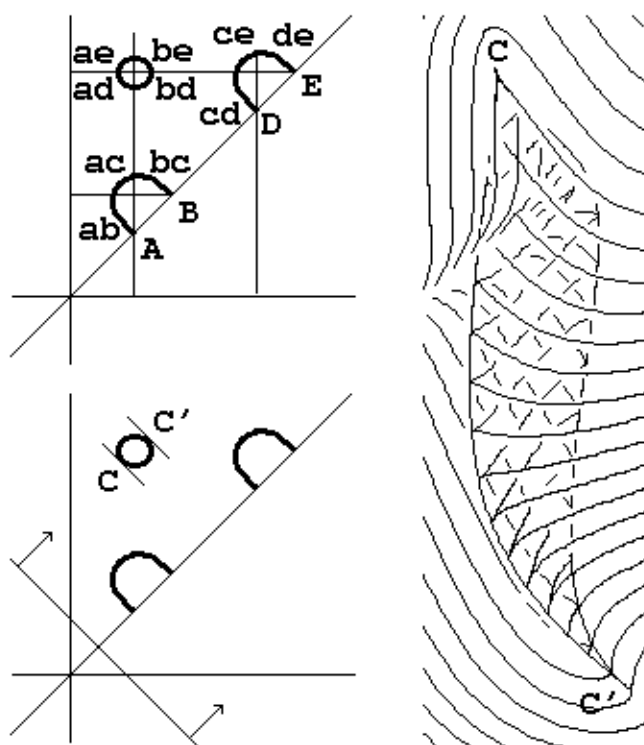
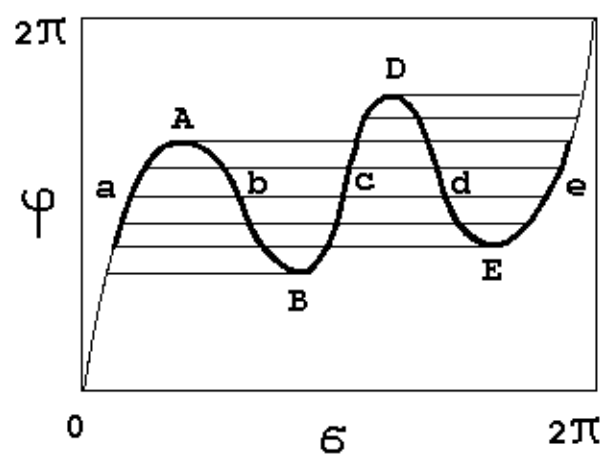


fig.12

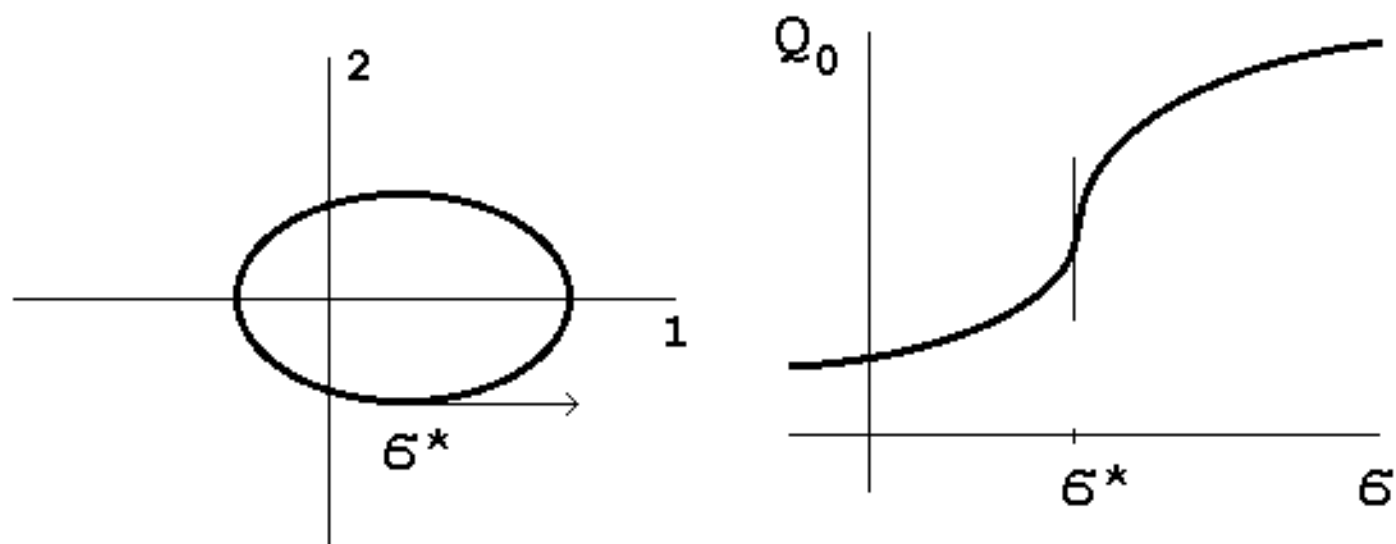


fig. 13

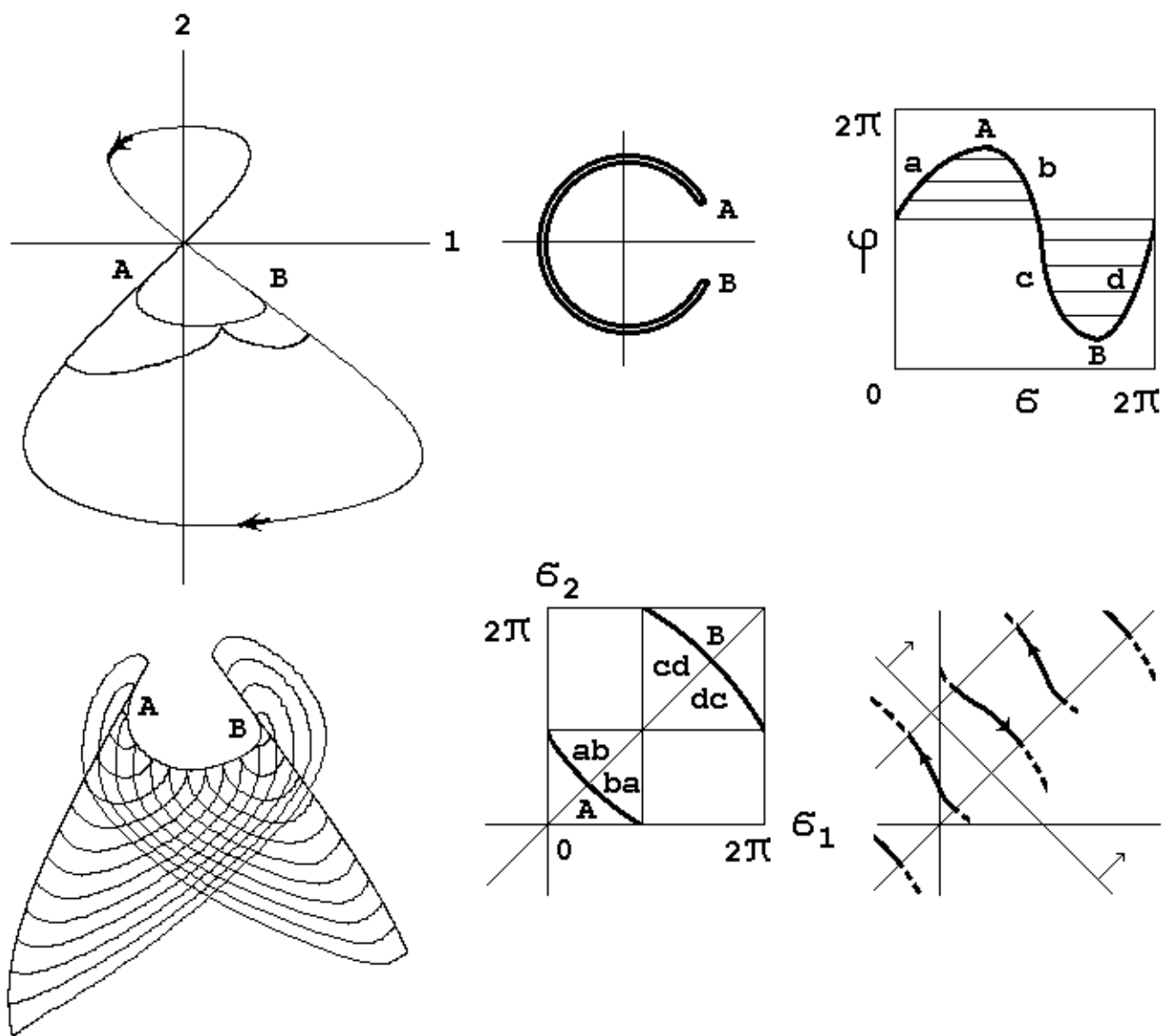


fig.14

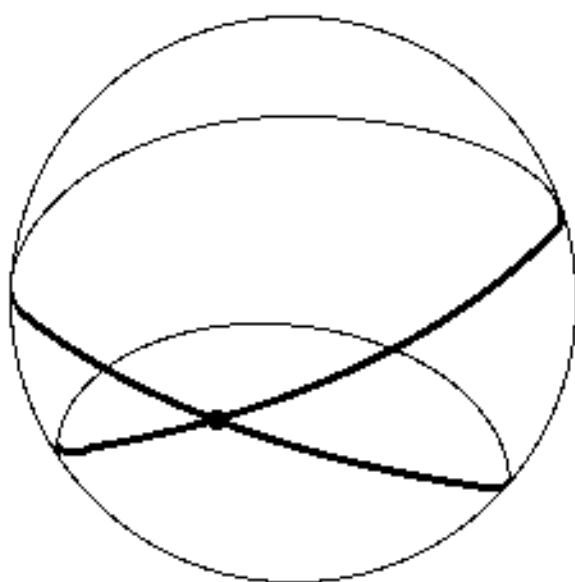


fig.15

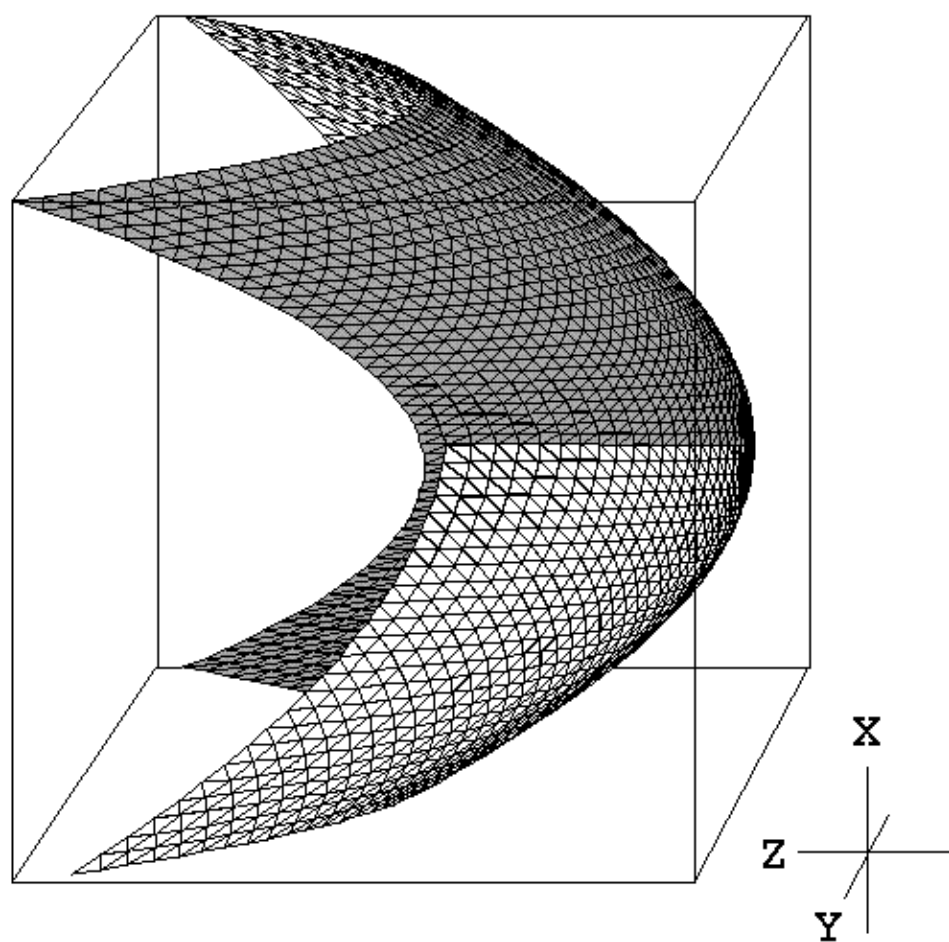


fig. 16

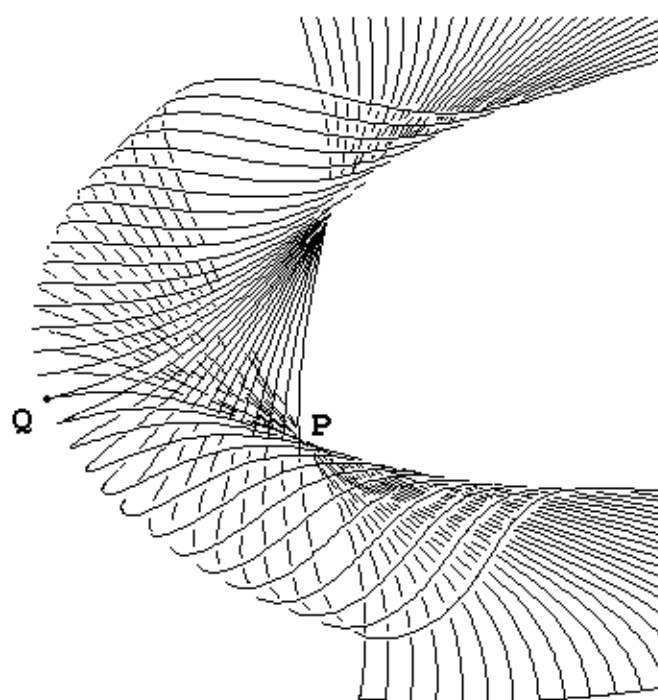


fig.17

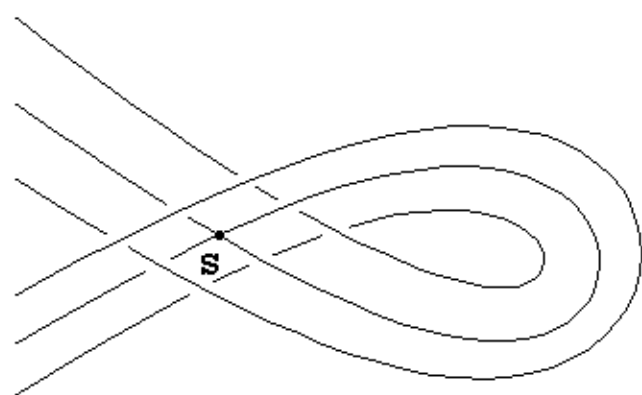


fig.18

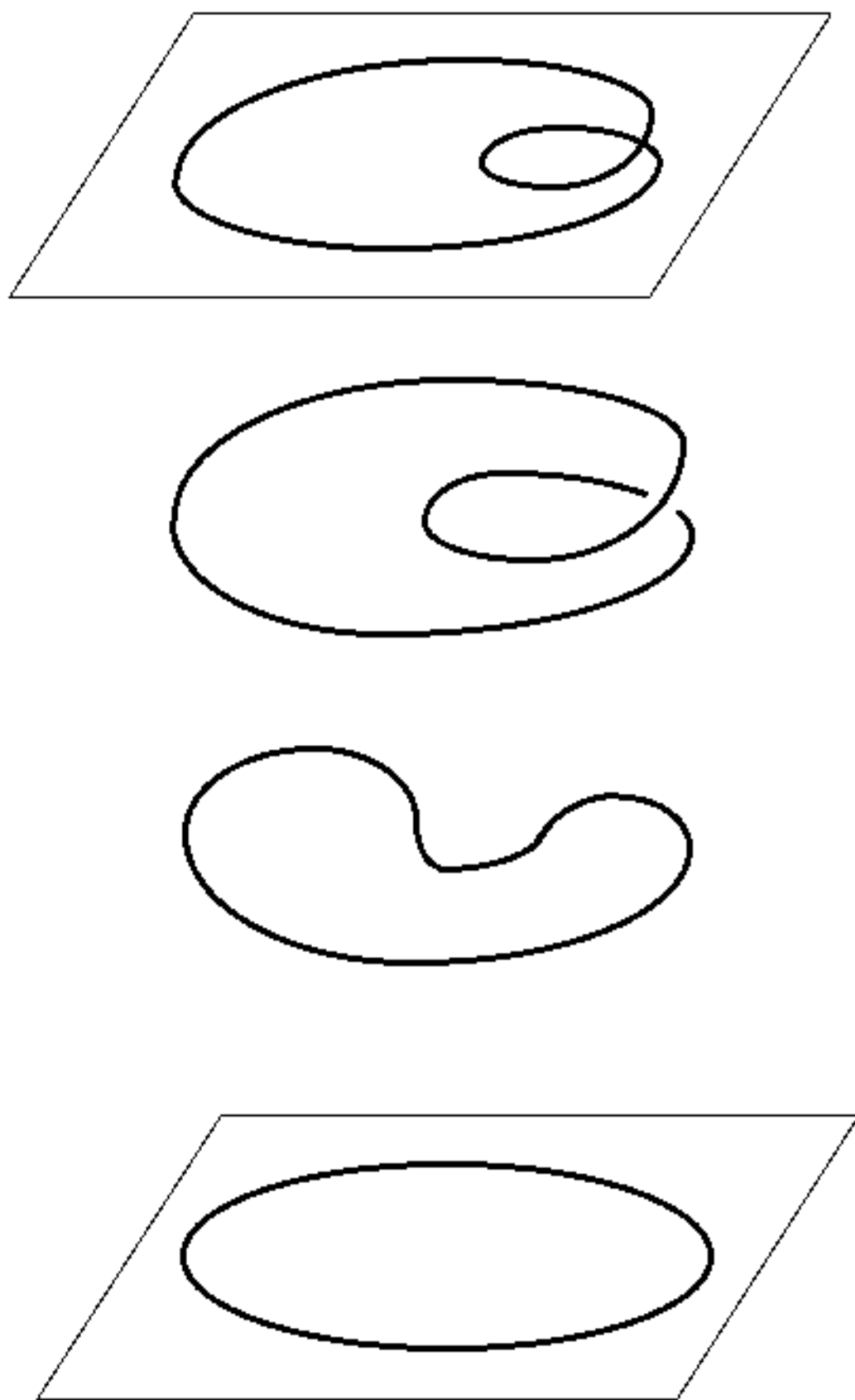


fig.19

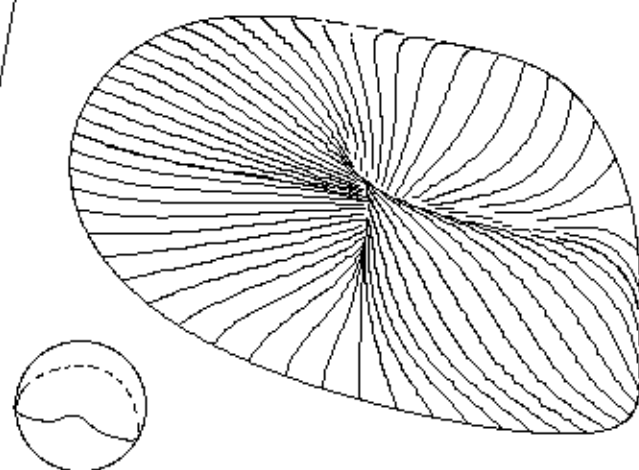
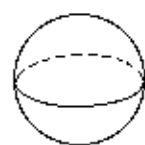
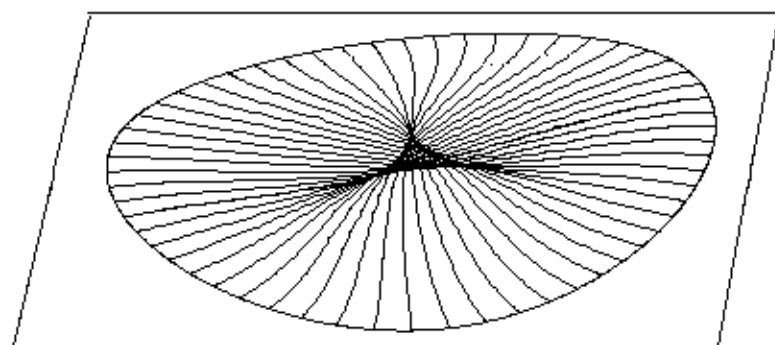
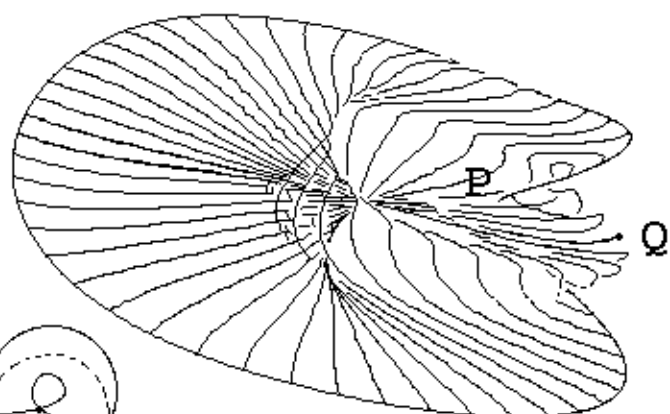
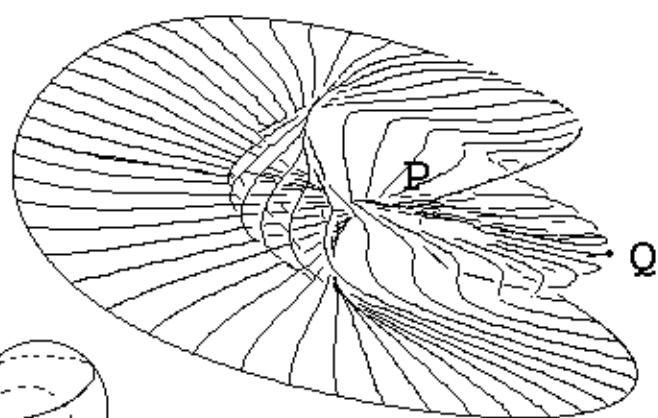
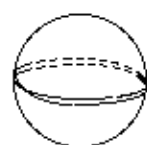
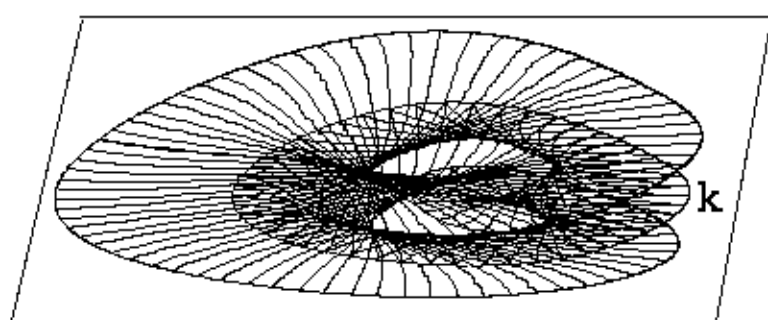


fig. 20

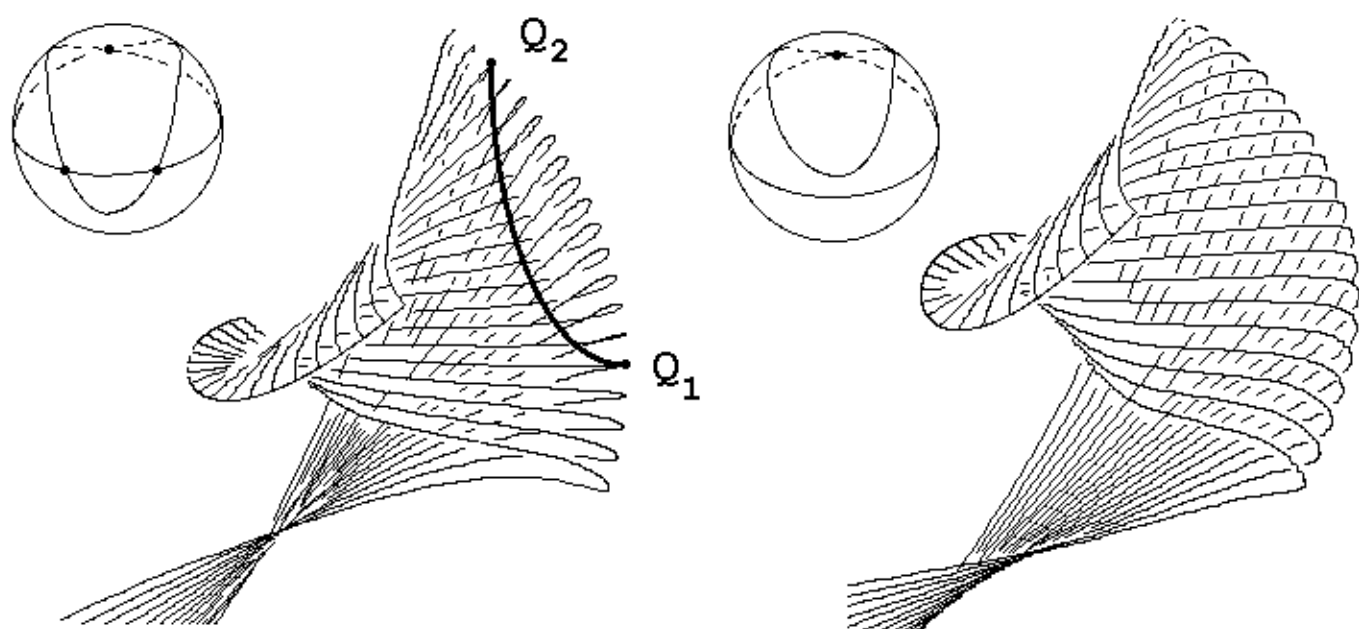


fig. 21

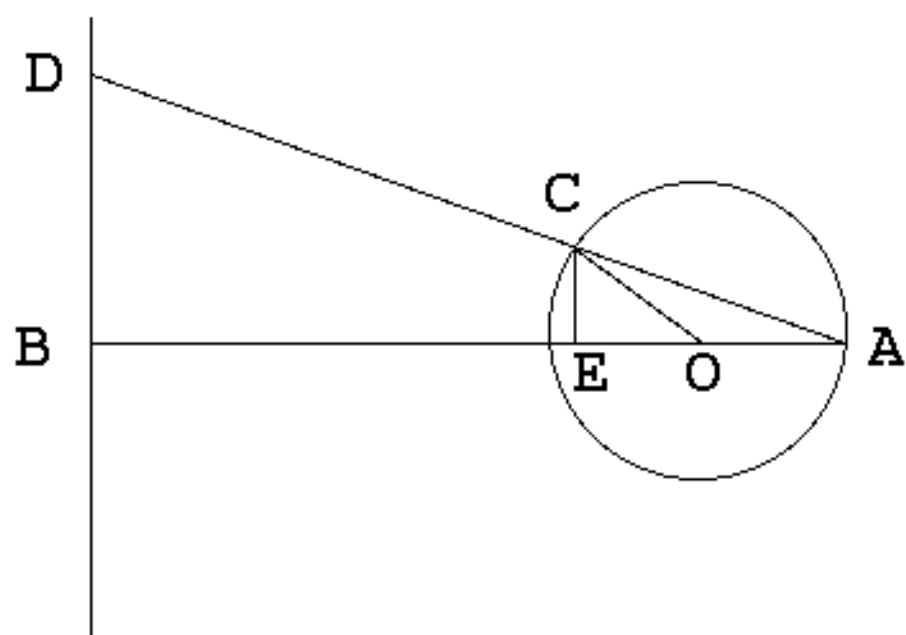
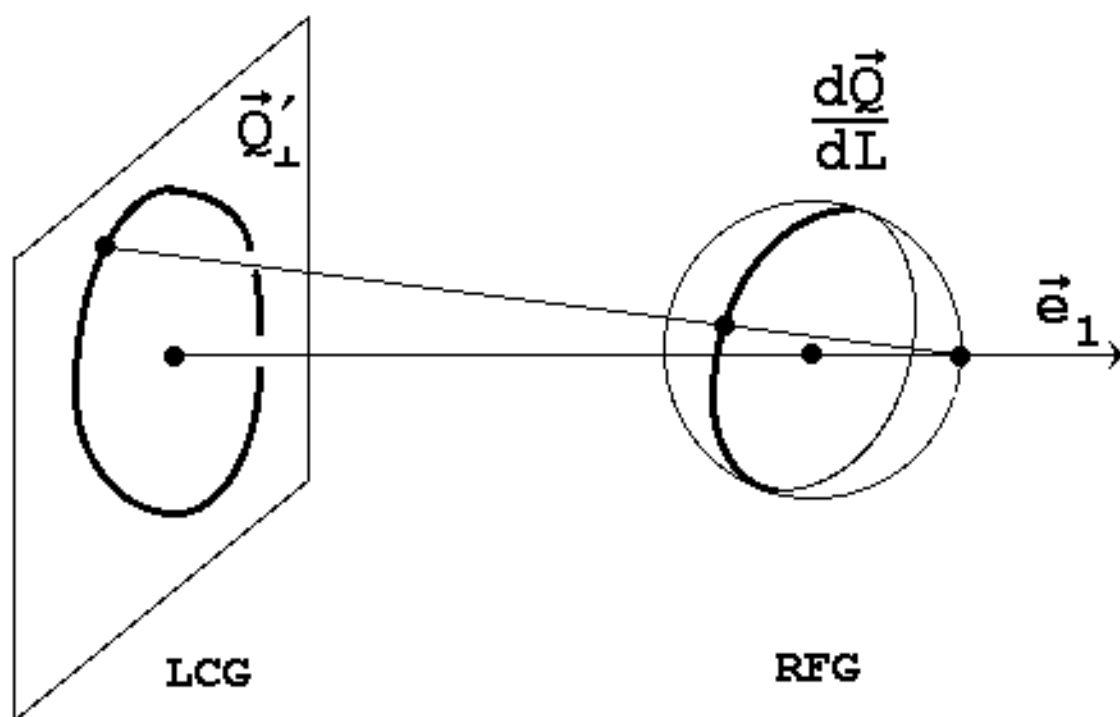


fig. 22

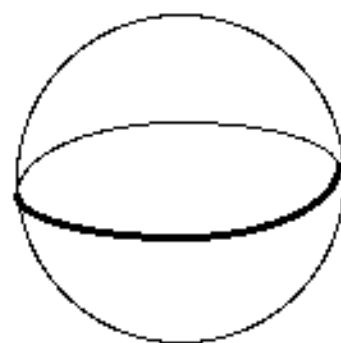
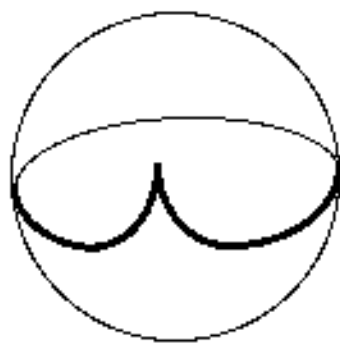
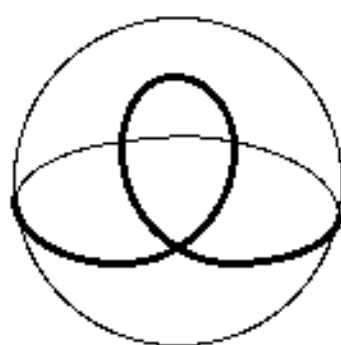


fig. 23

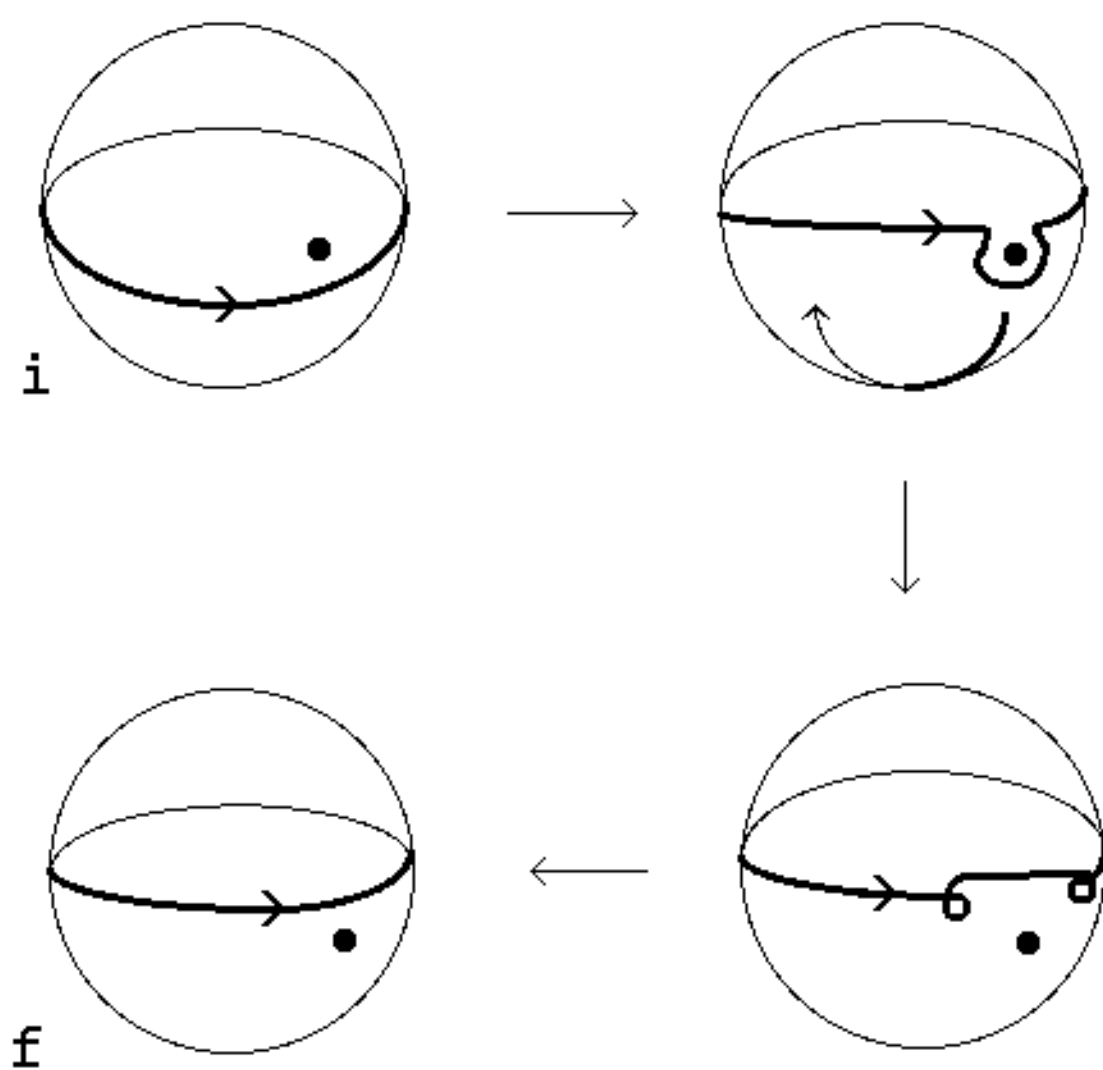


fig. 24

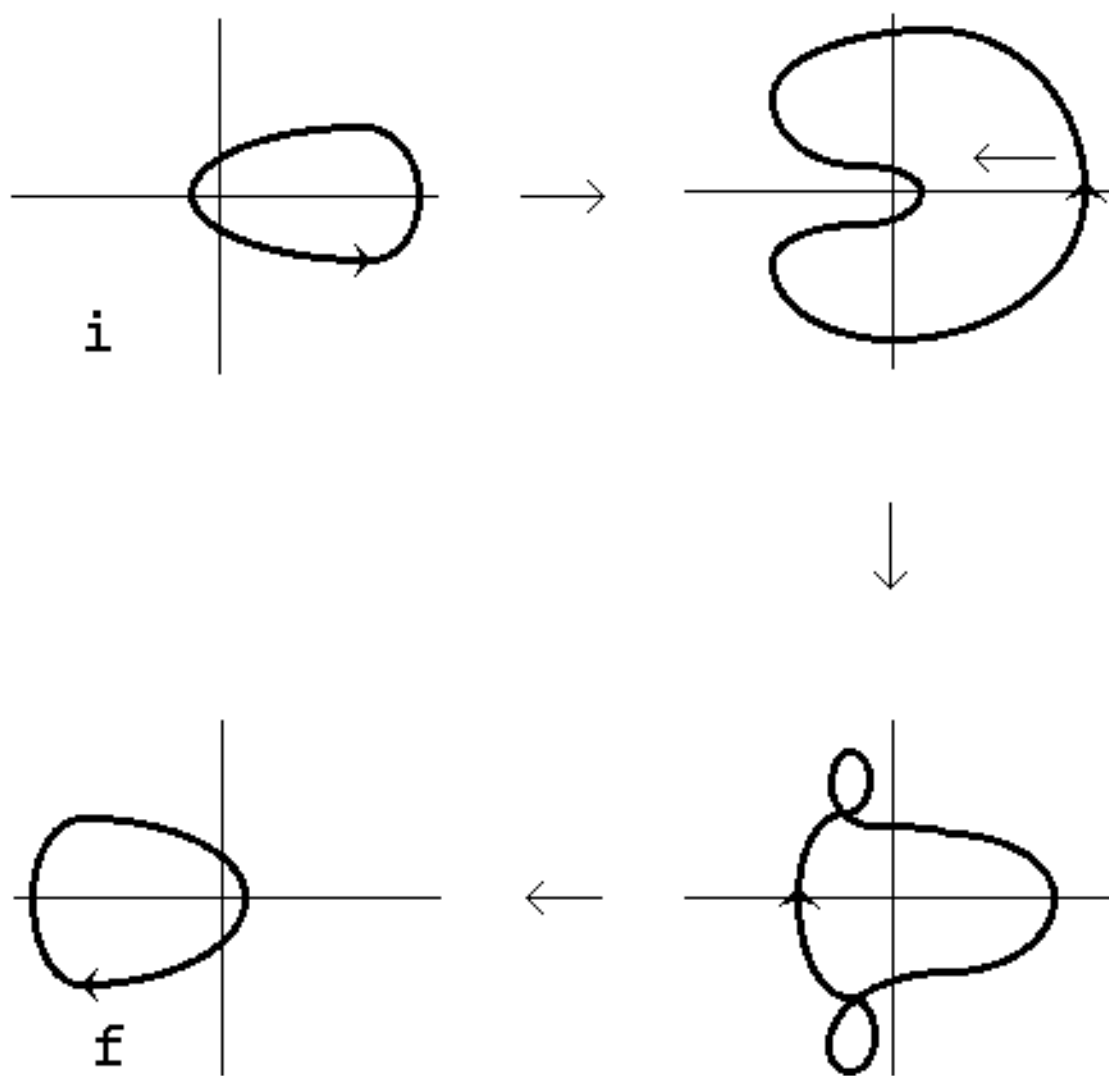


fig. 25

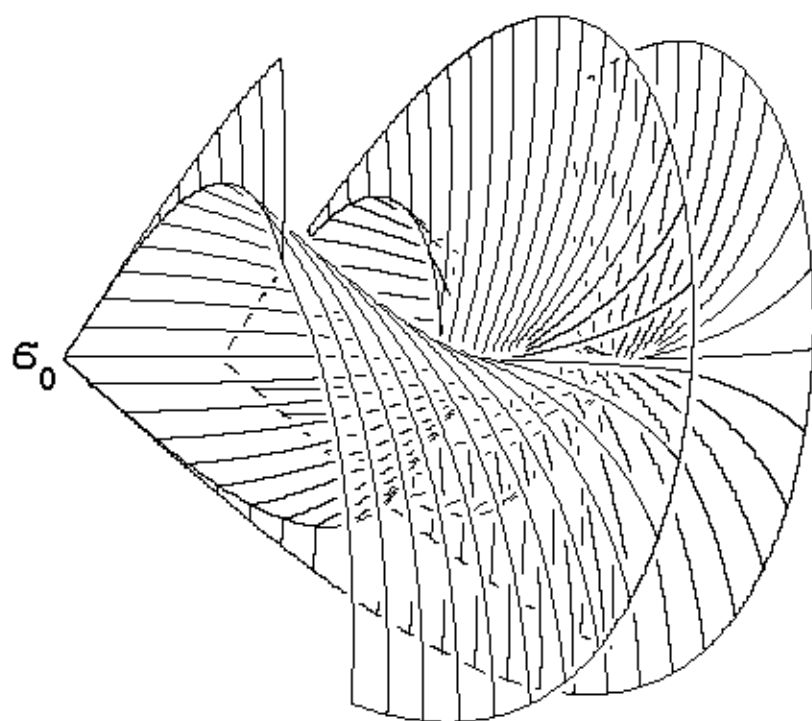


fig. 26

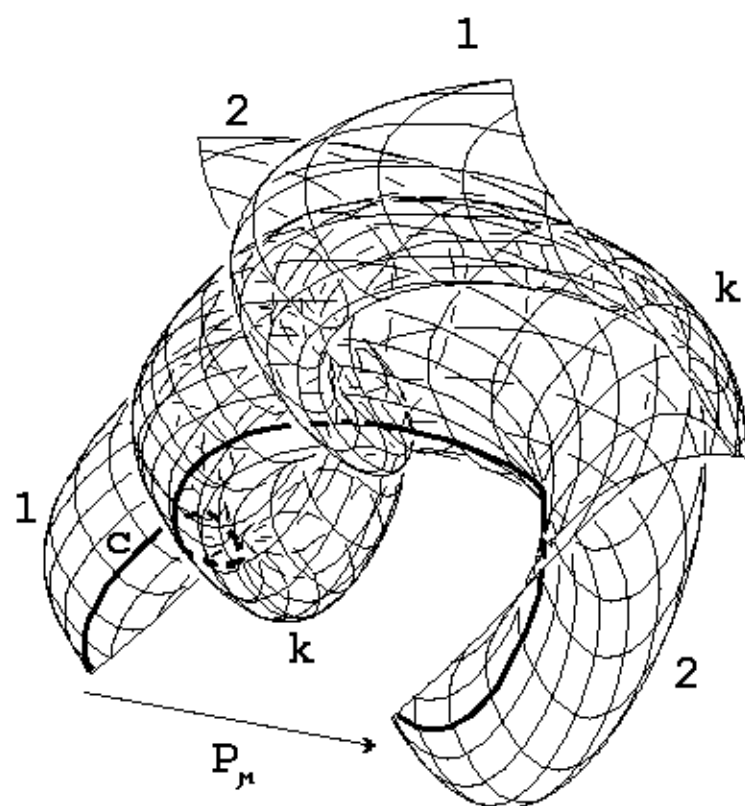


fig. 27

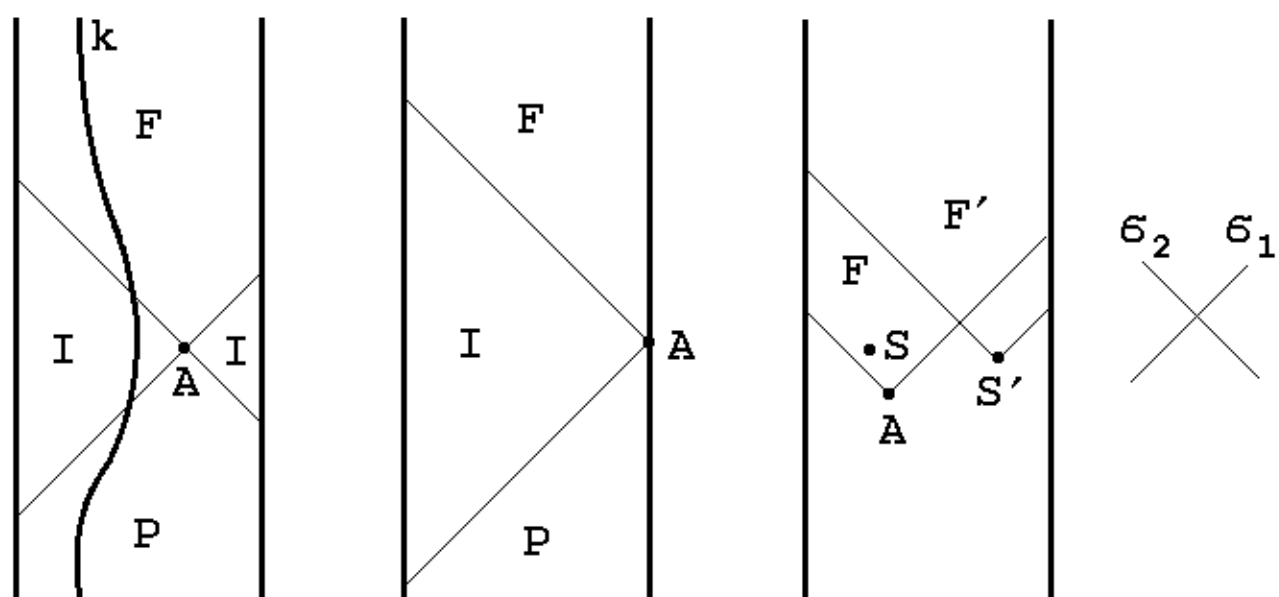


fig. 28